Steady free-surface potential flow due to a point source

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Consider the linearized free-surface potential flow generated by a point source of unit strength advancing with constant speed $\mathcal{U}$ along a straight path submerged a depth $\delta = D/L$ below the mean free-surface plane $z = 0$. Here L is an arbitrary reference length. This reference length may be chosen as $L = D$ or $L = \mathcal{U}^2/g$, which respectively yield $\delta = 1$ or $\delta = g\,D/\mathcal{U}^2$. The flow is observed from a moving system of coordinates attached to the source and thus appears steady. The z axis is vertical and points upward. The x axis is chosen parallel to the path of the moving source and points in the direction of motion of the source. The origin of the system of coordinates is chosen above the source, which is then located at the point $(0\,,0\,,-\delta)$.

BASIC FLOW REPRESENTATION

The disturbance velocity field $\vec{u}$ due to the unit source can be expressed as

$$\vec{u} = \vec{u}^{\,W} + \vec{u}^{\,S} + \vec{u}^{\,L} \tag{1}$$

where the simple-singularity component $\vec{u}^{\,S}$ corresponds to a simple Rankine singularity and the wave component $\vec{u}^{\,W}$ and the local component $\vec{u}^{\,L}$ account for free-surface effects.

The **wave component** $\vec{u}^{\,W}$ is given by

$$\left\{ \begin{matrix} u^W \\ v^W \\ w^W \end{matrix} \right\} = (\,1 - \text{sign}x\,)\,\frac{1}{\pi\,F^4}\int_0^\infty dt\;\alpha \left\{ \begin{matrix} \cos\varphi^x \cos\varphi^y \\ -t\,\sin\varphi^x \sin\varphi^y \\ \alpha\,\sin\varphi^x \cos\varphi^y \end{matrix} \right\} \exp\frac{(z-\delta)\,\alpha^2}{F^2} \tag{2a}$$

$$\text{where} \qquad \varphi^x = \frac{x\,\alpha}{F^2} \qquad\qquad \varphi^y = \frac{y\,t\,\alpha}{F^2} \qquad\qquad \text{with} \qquad \alpha = \sqrt{1+t^2} \tag{2b}$$

The **simple-singularity component** $\vec{u}^{\,S}$ is given by

$$\left\{ \begin{matrix} u^S \\ v^S \\ w^S \end{matrix} \right\} = \frac{1}{4\,\pi\,r^2} \left\{ \begin{matrix} x/r \\ y/r \\ (z+\delta)/r \end{matrix} \right\} \qquad\qquad \text{with} \qquad r = \sqrt{x^2+y^2+(z+\delta)^2} \tag{3}$$

The **local component** is the object of this note and is considered hereafter. Let

$$r' = \sqrt{x^2+y^2+(z-\delta)^2}$$

In the **farfield** $r'/F^2 \to \infty$, the local component $\vec{u}^{\,L}$ can be evaluated using an asymptotic approximation which can be obtained from *Bessho (1964)* and *Ponizy (1994)* :

$$4\,\pi\,F^4 \left\{ \begin{matrix} u^L \\ v^L \\ w^L \end{matrix} \right\} \sim \left\{ \begin{matrix} u_1^{LF} + u_2^{LF} \\ v_1^{LF} + v_2^{LF} \\ w_1^{LF} + w_2^{LF} \end{matrix} \right\} \quad \text{as} \quad r'/F^2 \to \infty \tag{4a}$$

where $\vec{u}_1^{LF}$ and $\vec{u}_2^{LF}$ are given by

$$\left\{ \begin{matrix} u_1^{LF} \\ v_1^{LF} \\ w_1^{LF} \end{matrix} \right\} = \left(\frac{F^2}{r'}\right)^2 \left\{ \begin{matrix} x/r' \\ y/r' \\ (z-\delta)/r' \end{matrix} \right\} \qquad \left\{ \begin{matrix} u_2^{LF} \\ v_2^{LF} \\ w_2^{LF} \end{matrix} \right\} = 4\left(\frac{F^2}{r'}\right)^3 \left\{ \begin{matrix} (A-B-1)\,x/r' \\ (A+B-3)\,y/r' \\ 1-\frac{3}{2}\,(\,Y_\infty^2+Z_\infty^2\,) \end{matrix} \right\} \tag{4b}$$

$$\text{with} \qquad Y_\infty = \frac{y}{r'} \qquad Z_\infty = \frac{\delta - z}{r'} \qquad B = \frac{Z_\infty}{1+Z_\infty}\left(\frac{3}{2} - \frac{Z_\infty/2}{1+Z_\infty}\right) \tag{4c}$$

$$A = \frac{Z_\infty}{1+Z_\infty}\left(\frac{9}{2} + \frac{Z_\infty}{1+Z_\infty}\left(\frac{1}{2}+\frac{3}{2}Z_\infty\right)\right) + \frac{Y_\infty^2}{(1+Z_\infty)^2}\left(4 + \frac{Z_\infty}{1+Z_\infty}\left(\frac{1}{2}+\frac{3}{2}Z_\infty\right)\right) \tag{4d}$$

In the **nearfield**, the local component $\vec{u}^{\,L}$ is given in *Noblesse (1978)* and *Ponizy (1994)* as

$$4\pi F^4 \begin{Bmatrix} u^L \\ v^L \\ w^L \end{Bmatrix} = \begin{Bmatrix} u_1^{LN} + \operatorname{sign}x\,(u_2^{LN} + u_3^{LN} + u_4^{LN}) \\ v_1^{LN} + v_2^{LN} + v_3^{LN} + v_4^{LN} \\ w_1^{LN} + w_2^{LN} + w_3^{LN} + w_4^{LN} \end{Bmatrix} \tag{5a}$$

The components $\vec{u}_1^{LN}$ and $\vec{u}_2^{LN}$ are singular at $r' = 0$ and are given by

$$\begin{Bmatrix} u_1^{LN} \\ v_1^{LN} \\ w_1^{LN} \end{Bmatrix} = -\left(\frac{F^2}{r'}\right)^2 \begin{Bmatrix} x/r' \\ y/r' \\ (z-\delta)/r' \end{Bmatrix} \qquad \begin{Bmatrix} u_2^{LN} \\ v_2^{LN} \\ w_2^{LN} \end{Bmatrix} = \frac{F^2}{r'} \begin{Bmatrix} 2\,Z_0 \\ 2\,Y_0\,Z_0 \\ 1+Y_0^2-Z_0^2 \end{Bmatrix} \tag{5b}$$

where $Y_0 = y/(r'+|x|)$ and $Z_0 = (\delta - z)/(r'+|x|$. The component $\vec{u}_3^{LN}$ is given by

$$\begin{Bmatrix} u_3^{LN} \\ v_3^{LN} \\ w_3^{LN} \end{Bmatrix} = \begin{Bmatrix} \ln(F^2/r') + \ln(1+Y_0^2+Z_0^2) - (\gamma + 1/2 - \ln 2) + (Y_0^2 - Z_0^2)/2 \\ Y_0\,(1+Y_0^2/3 - Z_0^2)/2 \\ -4/3 - Z_0\,(3 + Y_0^2 - Z_0^2/3)/2 \end{Bmatrix} \tag{5c}$$

where $\gamma \approx 0.577$ is Euler's constant. The component $\vec{u}_3^{LN}$ is $O(1)$ as $r' \to 0$. Finally, the component $\vec{u}_4^{LN}$ is $O(r'/F^2)$ as $r' \to 0$ and is given by

$$\begin{Bmatrix} u_4^{LN} \\ v_4^{LN} \\ w_4^{LN} \end{Bmatrix} = \frac{2}{\pi}\int_{-1}^{1} dt\,\sqrt{1-t^2} \begin{Bmatrix} \Re e\,\mathcal{F}(\mathcal{Z}) \\ t\,\Im m\,\mathcal{F}(\mathcal{Z}) \\ \sqrt{1-t^2}\,\Im m\,\mathcal{F}(\mathcal{Z}) \end{Bmatrix} \tag{5d}$$

$$\mathcal{F}(\mathcal{Z}) = \exp(\mathcal{Z})\,E_1(\mathcal{Z}) + \ln(\mathcal{Z}) + \gamma \qquad \mathcal{Z} = \left(\frac{z-\delta}{F^2}\sqrt{1-t^2} + \frac{y}{F^2}\,t + i\,\frac{|x|}{F^2}\sqrt{1-t^2}\right)\sqrt{1-t^2}$$

Here, $E_1(\mathcal{Z})$ is the exponential integral. $\mathcal{F}(\mathcal{Z})$ and consequently $\vec{u}_4^{LN}$ vanish as $r'/F^2 \to 0$.

Let the coordinates $(x, y, z-\delta)$ be defined in terms of the polar coordinates (r', θ, φ) :

$$\begin{Bmatrix} x = r'\cos\theta\cos\varphi \\ y = r'\cos\theta\sin\varphi \\ z - \delta = -r'\sin\theta \end{Bmatrix} \qquad \text{with} \qquad \begin{Bmatrix} 0 \le \theta \le \pi/2 \\ -\pi < \varphi \le \pi \end{Bmatrix}$$

The foregoing farfield and nearfield representations show that the local flow $F^4\,\vec{u}^{\,L}$ is a function of the 3 variables $(r'/F^2, \theta, \varphi)$ and that $4\pi\,(r')^2\,\vec{u}^{\,L}$ is $O(1)$ for $0 \le r'/F^2 \le \infty$.

GENERALIZED FLOW REPRESENTATION

Expressions (2) for the wave component $\vec{u}^{\,W}$ show that v^W and w^W vanish as $x \to 0$. However, u^W has a finite discontinuity at $x=0$. Specifically, we have

$$\pi F^4\,u^W \sim (1 - \operatorname{sign}x)\,J \;\text{ as } x \to 0 \qquad \text{with} \quad J = \int_0^\infty dt\;\alpha\,\cos\frac{y\,t\,\alpha}{F^2}\,\exp\frac{(z-\delta)\,\alpha^2}{F^2}$$

The nearfield representation (5) of the local component $\vec{u}^{\,L}$ shows that v^L and w^L are continuous at $x = 0$. However, u^L has a finite discontinuity at $x=0$. Specifically, we have

$$\pi F^4\,u^L \sim \operatorname{sign}x\,(u_2^{LN} + u_3^{LN} + u_4^{LN})/4 \qquad \text{as} \quad x \to 0$$

Continuity of the velocity $u^W + u^L$ at $x = 0$ yields $4\,J = [\,u_2^{LN} + u_3^{LN} + u_4^{LN}\,]_{x=0}$. The wave component $\vec{u}^{\,W}$ may be expressed in the form $\vec{u}^{\,W} = \vec{u}_*^{\,W} + \vec{u}_L^{\,W}$ where the component $\vec{u}_*^{\,W}$ is defined by (2) with sign$x$ in (2a) replaced by a function

$$\Theta \equiv \Theta(\frac{\sigma x}{F^2}) \quad \text{of the type} \qquad \mathrm{erf}(\frac{\sqrt{\pi}}{2}\frac{\sigma x}{F^2}) \qquad \tanh(\frac{\sigma x}{F^2}) \qquad \frac{x}{\sqrt{x^2 + (F^2/\sigma)^2}} \tag{6a}$$

Here, σ is an arbitrary positive real number. Note that we have

$$\Theta \sim \sigma x / F^2 = (\sigma\, r'/F^2\,)\, x/r' \qquad \text{as} \qquad x \to 0 \tag{6b}$$

The component $\vec{u}_L^{\,W}$ is given by

$$\begin{Bmatrix} u_L^W \\ v_L^W \\ w_L^W \end{Bmatrix} = (\,\Theta - \mathrm{sign}x\,)\,\frac{1}{\pi\,F^4}\int_0^\infty dt\;\alpha \begin{Bmatrix} \cos\varphi^x \cos\varphi^y \\ -t\,\sin\varphi^x \sin\varphi^y \\ \alpha\,\sin\varphi^x \cos\varphi^y \end{Bmatrix} \exp\frac{(z-\delta)\,\alpha^2}{F^2}$$

The component $\vec{u}_L^{\,W}$ represents a local flow, which can be grouped with $\vec{u}^{\,L}$. Thus, the flow representation (1) becomes $\vec{u} = \vec{u}_*^{\,W} + \vec{u}^{\,S} + (\,\vec{u}^{\,L} + \vec{u}_L^{\,W}\,)$. In the limit $x \to 0$, we have $v_L^W \to 0$, $w_L^W \to 0$, and $\pi\,F^4\,u_L^W \sim (\,\Theta - \mathrm{sign}x\,)\,J \sim (\,\Theta - \mathrm{sign}x\,)\,(\,u_2^{LN} + u_3^{LN} + u_4^{LN}\,)/4$. Thus, sign$x$ in (2a) and (5a) may be replaced by a function Θ of the type (6) if σ is sufficiently large. This substitution is used hereafter.

SIMPLE ANALYTICAL APPROXIMATION TO THE LOCAL FLOW

The farfield and nearfield representations (4) and (5) yield

$$4\,\pi\,F^4 \begin{Bmatrix} u^L \\ v^L \\ w^L \end{Bmatrix} \approx Q\left(\frac{F^2}{r'}\right)^2 \begin{Bmatrix} x/r' \\ y/r' \\ (z-\delta)/r' \end{Bmatrix} \quad \text{with} \quad \begin{Bmatrix} Q \to 1 \;\text{ as }\; r'/F^2 \to \infty \\ Q \to -1 \;\text{ as }\; r'/F^2 \to 0 \end{Bmatrix}$$

A simple function Q that satisfies these two limiting conditions is $Q = (\,r' - F^2\,)/(\,r' + F^2\,)$. Thus, the local flow $\vec{u}^{\,L}$ may be approximated as $\vec{u}^{\,L} \approx \vec{u}^{\,LA}$ with $\vec{u}^{\,LA}$ given by

$$4\,\pi\,F^4 \begin{Bmatrix} u^{LA} \\ v^{LA} \\ w^{LA} \end{Bmatrix} = -\left(\frac{F^2}{r'}\right)^2 \begin{Bmatrix} x/r' \\ y/r' \\ (z-\delta)/r' \end{Bmatrix} + \frac{2\,F^2/r'}{r'/F^2 + 1} \begin{Bmatrix} x/r' \\ y/r' \\ (z-\delta)/r' \end{Bmatrix}$$

This approximation to the local flow $\vec{u}^{\,L}$ is asymptotically equivalent to the first (dominant) terms in the farfield and nearfield approximations (4) and (5), and the *relative* error associated with the approximation $\vec{u}^{\,A}$ is $O(F^2/r')$ in the farfield and $O(r'/F^2)$ in the nearfield.

The second term in the approximation $\vec{u}^{\,LA}$ is $O(F^2/r')$ in the nearfield, like the second term $\vec{u}_2^{\,LN}$ defined by (5b) in the nearfield approximation (5a). The modified approximation given by

$$4\,\pi\,F^4 \begin{Bmatrix} u^{LA} \\ v^{LA} \\ w^{LA} \end{Bmatrix} = -\left(\frac{F^2}{r'}\right)^2 \begin{Bmatrix} x/r' \\ y/r' \\ (z-\delta)/r' \end{Bmatrix} + \frac{2\,F^2/r'}{(r'/F^2 + 1)^2} \begin{Bmatrix} x/F^2 \\ y/F^2 + Y_0\,Z_0 \\ (z-\delta)/F^2 + (1 + Y_0^2 - Z_0^2\,)/\,2 \end{Bmatrix} \tag{7a}$$

$$\text{where} \qquad r' = \sqrt{x^2 + y^2 + (z-\delta)^2} \qquad Y_0 = \frac{y}{r' + |x|} \qquad Z_0 = \frac{\delta - z}{r' + |x|} \tag{7b}$$

is asymptotically equivalent to the first term in the farfield approximation (4) and the first two terms in the nearfield approximation (5). In fact, (7) yields

$$4\,\pi\,F^4 \begin{Bmatrix} u^{LA} \\ v^{LA} \\ w^{LA} \end{Bmatrix} \sim \begin{Bmatrix} u_1^{LF} \\ v_1^{LF} \\ w_1^{LF} \end{Bmatrix} - 4\left(\frac{F^2}{r'}\right)^3 \begin{Bmatrix} x/r' \\ y/r' - Y_0\,Z_0\,/\,2 \\ (z-\delta)/r' - (1 + Y_0^2 - Z_0^2\,)/4 \end{Bmatrix} \quad \text{as} \quad \frac{r'}{F^2} \to \infty \tag{8a}$$

$$4\pi F^4 \begin{Bmatrix} u^{LA} \\ v^{LA} \\ w^{LA} \end{Bmatrix} \sim \begin{Bmatrix} u_1^{LN} \\ v_1^{LN} + v_2^{LN} \\ w_1^{LN} + w_2^{LN} \end{Bmatrix} + 2 \begin{Bmatrix} x/r' \\ y/r' - 2\,Y_0\,Z_0 \\ (z-\delta)/r' - (1+Y_0^2-Z_0^2) \end{Bmatrix} \quad \text{as} \quad \frac{r'}{F^2} \to 0 \tag{8b}$$

Thus, the *relative* error associated with the approximation (7) is $O(F^2/r')$ in the farfield and $O(r'/F^2)^2$ in the nearfield, as follows from (6b), (5a) and (5b).

For purposes of numerical evaluation, the approximation (7) can be expressed in the form

$$4\pi \begin{Bmatrix} u^{LA} \\ v^{LA} \\ w^{LA} \end{Bmatrix} = \begin{Bmatrix} P\,x \\ P\,y - Q\,Y\,Z \\ P\,(z-\delta) + \frac{1}{2}\,Q\,[\,1 + (Y+Z)(Y-Z)\,] \end{Bmatrix} \text{ with } \begin{Bmatrix} P = R\,(1-F^2/r'-S) \\ Q = r' R\,S \\ R = 1/[\,(r')^2(F^2+r')\,] \\ S = 2F^2/(F^2+r') \\ Y = y/(r'+|x|) \\ Z = (z-\delta)/(r'+|x|) \end{Bmatrix} \tag{9a}$$

Gauss integration rules can be used to integrate $\vec{u}^{\,LA}$ over a panel except if r' is of the order of the panel size. In this case, $\vec{u}^{\,LA}$ can be expressed as

$$4\pi \begin{Bmatrix} u^{LA} \\ v^{LA} \\ w^{LA} \end{Bmatrix} = \frac{-1}{(r')^2} \begin{Bmatrix} x/r' \\ y/r' \\ (z-\delta)/r' \end{Bmatrix} + \frac{1}{F^2 r'} \begin{Bmatrix} 0 \\ -2\,\overline{Y}\,\overline{Z} \\ 1+\overline{Y}^2-\overline{Z}^2 \end{Bmatrix} + \frac{2}{F^4} \begin{Bmatrix} u_* \\ v_* \\ w_* \end{Bmatrix} \tag{9b}$$

where $\overline{Y}$ and $\overline{Z}$ stand for mean values of Y and Z over the panel, and $\vec{u}_*$ is given by

$$\begin{Bmatrix} u_* \\ v_* \\ w_* \end{Bmatrix} = \frac{1}{(1+\rho)^2} \begin{Bmatrix} x/r' \\ y/r' + 2\,\overline{Y}\,\overline{Z}\,(1+\rho/2) + (\,\overline{Y}\,\overline{Z} - Y Z\,)/\rho \\ (z-\delta)/r' - (1+\overline{Y}^2-\overline{Z}^2)\,(1+\rho/2) + \frac{1}{2}\,(\,Y^2-Z^2+\overline{Z}^2-\overline{Y}^2\,)/\rho \end{Bmatrix} \tag{9c}$$

with $\rho = r'/F^2$. The first two terms on the right of (9b) are singular as $r' \to 0$ but can be integrated analytically. The term $\vec{u}_*$ is finite as $r' \to 0$ and Gauss integration can be used. Thus, the component $\vec{u}^{\,LA}$ can be integrated over a panel in a simple and accurate manner.

CONCLUSION

The analytical approximation $\vec{u}^{\,LA}$ given by (9) makes it possible to evaluate the flow due to a singularity distribution in a simple and highly efficient manner. Practical applications to the slender-ship source distribution defined in *Noblesse (1983)* show that the approximation (9) is sufficient for many purposes. The approximation (7) also provides a useful starting point to develop accurate approximations to the local flow $\vec{u}^{\,L}$ by expressing $\vec{u}^{\,L}$ in the form

$$\vec{u}^{\,L} = \vec{u}^{\,LA} + \vec{u}^{\,LR}$$

The remainder $\vec{u}^{\,LR} = \vec{u}^{\,L} - \vec{u}^{\,LA}$ is $O(1)$ as $r' \to 0$ and $O(1/r')^3$ as $r' \to \infty$. This remainder can be evaluated using table interpolation as in *Ponizy (1994)*, or an analytical approximation based on a composite of the farfield and nearfield approximations (8), (4) and (5).

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