TRAPPED MODES ABOVE A DIE OSCILLATING ON THE BOTTOM OF A WAVE CHANNEL

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1. Introduction

The problem of body's oscillation in water confound in a channel is often considered in the framework of the shallow water theory. However there are cases when trapped modes exist if the linear finite-depth solution applies and do not exist in the model by the shallow water theory. Trapped modes represent a localized oscillation of finite energy which does not propagate away to infinity.

In the paper [1] the existence of trapped modes travelling along the plate placed on the bottom of the channel was established. This problem was reduced to the two-dimensional boundary value problem, and its continuous spectrum is above a non-zero cut-off frequency. It was shown that at least one trapped mode exists occurring at frequency which is less then the cut-off value. The same result was obtained for shallow water. The case when frequency is above the cut-off was not discussed.

If there is no cut-off value in the problem, the model of shallow water fails to prove the existence of trapped modes. For example Zilman & Miloh [2] showed that oscillation of a buoyant circular plate in shallow water leads to formation only outgoing surface waves. On the other hand, for deep water the same velocity potential defined with the help of the Green function [3] gives both standing and outgoing waves, and the evidence of the trapped modes existence can be obtained numerically.

In the note we consider a circular die on the bottom of a channel. The simple geometry of the die allows us to find the analytic solution of the problem. The cut-off frequency of the boundary-value problem is zero and there exist only progressive waves in shallow water. The purpose of this work is to establish the existence of trapped modes occurring at frequencies which are point eigenvalues of the problem embedded in the continuous spectrum.

2. Formulation

We consider a three-dimensional channel of constant depth h occupied by an inviscid and incompressible fluid. The moving part of the bottom is modelled by a circular die of radius a. Cartesian coordinates are chosen so that the origin is in the position of the center of the die at rest, y-axis points vertically upwards and (x_1, x_2) are in plane of the unperturbed bottom.

It is assumed that the motion is simple harmonic in the time and has the radian frequency ω. The vertical displacement of the die can be written as $\mathrm{Re}\{\zeta_0 e^{-\imath\omega t}\}$ where ζ_0 is the amplitude of die oscillations. The velocity potential which describes the fluid motion is given by $\mathrm{Re}\{\varphi(x_1, x_2, y)e^{-\imath\omega t}\}$, where φ satisfies Laplace's equation

$$\nabla^2\varphi = 0, \quad \text{in the fluid}, \tag{1}$$

and the boundary conditions

$$\varphi_y - \frac{\omega^2}{g}\varphi = 0 \quad \text{on} \quad y = h, \tag{2}$$

$$\frac{\partial \varphi}{\partial y} = -\mathrm{i}\omega\zeta_0 \quad \text{on} \quad |x| < a, \quad y = 0, \tag{3}$$

$$\frac{\partial \varphi}{\partial y} = 0 \quad \text{on} \quad |x| > a, \quad y = 0, \tag{4}$$

where g is the acceleration due to gravity and $|x| = (x_1^2 + x_2^2)^{1/2}$. If trapped modes are sought, then the radiation condition is replaced by

$$\varphi \to 0 \quad \text{as} \quad |x| \to \infty. \tag{5}$$

The displacement of the die ζ_0 is determined by the equation

$$(\overline{C} - M\omega^2)\zeta_0 = -\frac{\mathrm{i}\rho\omega}{\pi a^2}\int_{S_a} \varphi \mathrm{d}x_1 \mathrm{d}x_2, \quad y = 0, \tag{6}$$

where ρ is the fluid density, M is the die mass per unit area, C is the elastic foundation rigidity, $\overline{C} = C - \rho g$, $S_a = \{|x| < a, \quad y = 0\}$ is the wetted surface of the die.

3. The velocity potential and trapped frequencies

It is convenient to use cylindrical coordinates (r, θ, y) defined by

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \quad r > 0.$$

A trapped mode solution of the problem (1)–(6) can be constructed with the help of the ring Green function which gives the potential of a ring source of radius ϱ in the bottom of the channel. The representation of the ring Green function can be found in [4] and [5]. Then the velocity potential has the form

$$\varphi(r, y) = 4\pi\mathrm{i}\zeta_0\omega \left\{ \mathrm{i}C_0 \int_0^a \varrho J_0(k_0 r_<) H_0^{(1)}(k_0 r_>) \mathrm{d}\varrho + \right.$$
$$\left. 2\sum_{m=1}^{\infty} C_m \int_0^a \varrho I_0(k_m r_<) K_0(k_m r_>) \mathrm{d}\varrho \right\} \tag{7}$$

where $J_0, \ I_0, \ K_0, \ H_0^{(1)}$ denote the standard Bessel, modified Bessel and Hankel functions of zero order,

$$r_< = \min\{r, \varrho\}, \quad r_> = \max\{r, \varrho\}$$

and

$$C_0 = \frac{2\pi(k_0^2 - \nu^2)}{h(k_0^2 - \nu^2) + \nu}\cosh k_0 y, \quad C_m = \frac{(k_m^2 + \nu^2)}{h(k_m^2 + \nu^2) - \nu}\cos k_m y,$$

where

$$k_0, \ \pm\mathrm{i}k_1, \ \pm\mathrm{i}k_2, ..., \ \pm\mathrm{i}k_n, ...$$

is a sequence of roots of the dispersion relation

$$k\tanh kh = \nu; \tag{8}$$

the parameter $\nu = \omega^2/g$.

Hence, (7) gives an expansion of the velocity potential in terms of the cylindrical waves. The behaviour of these waves is clear from the asymptotic representation of Bessel functions. At large distance the contribution by the first term in (7) is an outgoing progressive wave as a result of expansion of the Hankel function. The waves described by the second term in (7) behave like standing waves which decay as $r \to \infty$. The outgoing wave in $r > a$ is annulled by taking $k_0 a$ to satisfy

$$J_1(k_0 a) = 0,$$

that is $k_0 a = j_n$ $(n = 1, 2, ...)$, where j_n are zeros of the Bessel function J_1. Thus, from (8) we find the sequence of frequencies

$$\omega_n = \left(\frac{j_n g}{a} \tanh\left(\frac{j_n h}{a}\right)\right)^{1/2}, \quad n = 1, 2, ... \tag{9}$$

Such that for $\omega = \omega_n$ outgoing wave becomes zero in $r > a$ and radiation condition (5) is fulfilled.

Now the aim is to find the set of the parameters $\{a, h, M, C\}$ for which the trapped frequency is fundamental one. Then (7) gives the trapped modes solution for $\omega = \omega_n$.

Substituting the formula (7) for the velocity potential into the integral in (6) we get for $\omega = \omega_n$

$$\overline{C}' - M' {\omega'_n}^2 = R'_g + R'_{in}, \quad n = 1, 2, ... \tag{10}$$

where we use the nondimensional parameters $\overline{C}' = \overline{C}/\rho g$, $M' = M/\rho a$, $R' = R/\rho g$, $\omega'_n = j_n \tanh j_n \delta$, $\delta = h/a$. The terms R'_g and R'_{in} in (10) are defined as follows

$$R'_g = -\frac{4 \tanh j_n \delta}{2 j_n \delta + \sinh 2 j_n \delta}, \quad n = 1, 2, ..., \tag{11}$$

$$R'_{in} = 4 j_n \tanh j_n \delta \sum_{m=1}^{\infty} \frac{1 - 2 I_1(k_{mn} a) K_1(k_{mn} a)}{k_{mn} a (2 k_{mn} h + \sin 2 k_{mn} h)}, \quad n = 1, 2, ... \tag{12}$$

and $\pi(m - \frac{1}{2})/h < k_{mn} < \pi m/h$, $(n = 1, 2, ...)$. The expression of the right-hand side of (10) $R' = R'_g + R'_{in}$ is a function of δ and represents the dynamical reaction of the fluid on die oscillations. The first term $R'_g < 0$ is a result of the gravity action caused by the standing waves and $R'_g = -1 + O(\delta)$ as $\delta \to 0$, and decays as $\delta \to \infty$. The second term $R'_{in} > 0$ is induced by inertial force of the fluid and $R'_{in} = O(\delta^2)$ as $\delta \to 0$ and $R'_{in} = j_n M'_{\rho_n}$ $(n = 1, 2, ...)$ as $\delta \to \infty$, where M'_{ρ_n} is the sum of the series (12). The constant M'_{ρ_n} $(n = 1, 2, ...)$ is the added mass of the die oscillating with n-th trapped frequency. Thus $R'(\delta)$ is a continuous function with alternating sings and R' tends to -1 as $\delta \to 0$ and R' equals to $j_n M'_{\rho_n}$, $(n = 1, 2, ...)$ as $\delta \to \infty$.

Using (10) to evaluate $\overline{C}'$ and taking into account the behaviour of $R'(\delta)$ we can formulate the following statement.

For any n, $\delta > 0$, and $M' > 0$ a finite solution $C' > 0$ of the equation (10) can be found where ω'_n (9) is the fundamental frequency.

Thus there exists the sequence $\omega_n \to \infty$ $(n = 1, 2, ...)$ given by (9), whose elements are point eigenvalues of the problem (1)–(6) embedded in its continuous spectrum. The trapped mode solutions $\varphi(\omega_n)$ are defined by (7).

4. Conclusion

The trapped modes solutions embedded in the continuous spectrum have been constructed for the circular die oscillating on the bottom of deep water. In the case when the geometry of the die is arbitrary the solution of the boundary value problem is a subject of numerical investigation. We can apply the same methods in order to prove the existence of trapped modes. The condition of the outgoing wave destruction has the form

$$\int_S H_0^{(1)}(k|x-\xi|)\mathrm{d}S = 0 \quad \text{as} \quad |x| \to \infty$$

where $x = (x_1, x_2)$, $\xi = (\xi_1, \xi_2)$ and S is the die wetted surface. Defining k from the last equation we find a spectrum of trapped frequencies, which can be written as

$$\omega_{tr} = (kg \tanh kh)^{1/2}$$

For the existence of trapped modes it is necessary to find parameters of the problem resolving the frequency equation

$$\overline{C} = \left[M + \frac{2\rho}{S} \int_S \int_S G(x, y; \xi)\mathrm{d}S\mathrm{d}S \right] \omega_{tr}^2$$

Here $G(x, y; \xi)$ is the Green function given in [3] which describes the velocity potential of a source placed at a point $(\xi, 0)$.

Acknowledgments. The authors are indebted to Prof. N.G. Kuznetsov and Prof. T. Miloh for substantive discussions and assistance.

References

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