SCATTERING OF OBLIQUE WAVES IN A TWO-LAYER FLUID

J.R. Cadby & C.M.Linton
Loughborough University, Loughborough, Leicestershire, LE11 3TU, U.K.

Introduction

The research presented here investigates the interaction of oblique waves with a horizontal cylinder in a two-layer fluid consisting of a layer of finite thickness bounded above by a free surface and below by a fluid of greater density and infinite depth. The free surface and the interface between the two fluids provide two surfaces on which waves can exist. The case of normal incidence was considered previously by Linton and McIver (1995) who were interested in such interactions following proposals to build submerged floating tunnels across Norwegian fjords. Such fjords typically consist of a layer of fresh water above a deep expanse of salt water.

Formulation

Cartesian coordinates are chosen such that the (x, y)-plane coincides with the undisturbed interface between the two fluids. The z-axis points vertically upwards with $z = 0$ and $z = d$ corresponding to the undisturbed interface and free surface respectively. The upper fluid, $0 < z < d$, will be referred to as region I and have density ρ^I, whilst the lower fluid, $z < 0$, will be referred to as region II and have density ρ^{II}. We also define $\rho = \rho^I/\rho^{II} < 1$. Under the usual assumptions of linear water wave theory we can write the velocity potential as $\Re\{\phi(x, z)e^{ily}e^{-i\omega t}\}$ where

$$(\nabla^2 - l^2)\phi = 0 \qquad \text{in the fluid,} \tag{1}$$

$$\phi_z^I = \phi_z^{II} \qquad \text{on } z = 0, \tag{2}$$

$$\rho(\phi_z^I - K\phi^I) = \phi_z^{II} - K\phi^{II} \qquad \text{on } z = 0, \tag{3}$$

$$\phi_z^I = K\phi^I \qquad \text{on } z = d, \tag{4}$$

and $K = \omega^2/g$. The dispersion relation is given by

$$(u - K)(K(\sigma + e^{-2ud}) - u(1 - e^{-2ud})) = 0, \tag{5}$$

where $\sigma = (1 + \rho)/(1 - \rho)$. It follows that either $u = K$ or $u = k > K$ where

$$K(\sigma + e^{-2kd}) = k(1 - e^{-2kd}). \tag{6}$$

The potential of an incident plane wave of wavenumber K making an angle α_{inc} with the positive x-axis takes the form

$$\phi_{\text{inc}} = e^{iKx\cos\alpha_{\text{inc}}}e^{Kz}, \tag{7}$$

from which

$$l = K\sin\alpha_{\text{inc}} \tag{8}$$

and hence we clearly must have $l < K$. For incident wavenumber k we simply have to replace K with k in (8) and such waves can exist provided $l < k$. A general scattering potential has the far-field behaviour described by

$$\phi^I \sim A^\pm e^{\pm i\beta x}e^{Kz} + B^\pm e^{\pm ibx}g(z) + C^\pm e^{\mp i\beta x}e^{Kz} + D^\pm e^{\mp ibx}g(z), \tag{9}$$

$$\phi^{II} \sim A^\pm e^{\pm i\beta x}e^{Kz} + B^\pm e^{\pm ibx}e^{kz} + C^\pm e^{\mp i\beta x}e^{Kz} + D^\pm e^{\mp ibx}e^{kz}, \tag{10}$$

as $x \to \pm\infty$, where $\beta = \sqrt{K^2 - l^2}$ and $b = \sqrt{k^2 - l^2}$. Equations (9) & (10) can be characterised by

$$\phi \sim \{A^-, B^-, C^-, D^-; A^+, B^+, C^+, D^+\}. \tag{11}$$

For incident waves of wavenumber K we have $\beta = K\cos\alpha_{\text{inc}}$ and $b = \sqrt{k^2 - K^2\sin^2\alpha_{\text{inc}}}$ which are real for all K and $0 < \alpha_{\text{inc}} < \pi/2$ since $K < k$. For the case of an incident wave of wavenumber k we have $\beta = \sqrt{K^2 - k^2\sin^2\alpha_{\text{inc}}}$ and $b = k\cos\alpha_{\text{inc}}$. In this case, given α_{inc} there is a critical value of K for which $\beta = 0$. For values of K less than this critical value β is complex and hence scattered waves of wavenumber K do not exist.

Using Green's theorem and (3) we can derive the general identity

$$\begin{aligned}\int_B \left(\phi_i \frac{\partial\phi_j}{\partial n} - \phi_j \frac{\partial\phi_i}{\partial n}\right) ds =& J_K(A_i^+C_j^+ - C_i^+A_j^+ + A_i^-C_j^- - C_i^-A_j^-) \\ &+ J_k(B_i^+D_j^+ - D_i^+B_j^+ + B_i^-D_j^- - D_i^-B_j^-), \qquad (12)\end{aligned}$$

where

$$J_K = i\beta\left[\frac{1}{K} + 2\rho\int_0^d e^{2Kz}dz\right], \qquad J_k = ib\left[\frac{1}{k} + 2\rho\int_0^d [g(z)]^2 dz\right], \qquad (13)$$

and B is the set of body boundaries which are assumed for simplicity to all lie in the lower fluid. If we consider the scattering of waves by a fixed obstacle then in general there are two problems. These are the scattering of an incident wave of wavenumber K, which we shall refer to as problem 1; and the scattering of an incident wave of wavenumber k (problem 2). For each of these problems we use R and T to represent reflection and transmission coefficients corresponding to waves of wavenumber K and r and t are used for waves of wavenumber k. The problems are thus characterised, in the notation of (11), by

$$\phi_1 \sim \{R_1, r_1, 1, 0; T_1, t_1, 0, 0\}, \qquad (14)$$

$$\phi_2 \sim \{R_2, r_2, 0, 1; T_2, t_2, 0, 0\}. \qquad (15)$$

Applying (12) to $\phi_i, \overline{\phi_i}, i = 1, 2$, we obtain identities representing energy conservation, namely

$$|R_1|^2 + |T_1|^2 + J(|t_1|^2 + |r_1|^2) = 1, \qquad (16)$$

$$|R_2|^2 + |T_2|^2 + J(|t_2|^2 + |r_2|^2) = J. \qquad (17)$$

where $J = J_k/J_K$.

Circular cylinder in lower fluid layer

Let us consider the case of an infinite cylinder of radius a in the lower fluid centred at $z = f < 0$ with its generator parallel to the y-axis. The scattering problem for such a geometry can be solved using multipole expansions. Multipoles are singular solutions of (1) which satisfy all the boundary conditions of the problem except that on the cylinder. Symmetric and antisymmetric multipoles can be constructed for this problem and when expanded about $z = f$ they take the form

$$\phi_m^s = K_m(lr)\cos m\theta + \sum_{n=0}^{\infty} A_{mn}^s I_n(lr)\cos n\theta, \qquad (18)$$

$$\phi_m^a = K_m(lr)\sin m\theta + \sum_{n=1}^{\infty} A_{mn}^a I_n(lr)\sin n\theta, \qquad (19)$$

where

$$A_{mn}^s = (-1)^{m+n}\epsilon_n \fint_0^\infty \cosh mu \cosh nu \; e^{2lf\cosh u} C_L(u)du, \qquad (20)$$

$$A_{mn}^a = (-1)^{m+n} 2 \fint_0^\infty \sinh mu \sinh nu \; e^{2lf\cosh u} C_L(u)du. \qquad (21)$$

Here $\epsilon_0 = 1, \epsilon_n = 2, n \geq 1$,

$$C_L(u) = \frac{(l\cosh u + K)[(l\cosh u + K\sigma)e^{-2ld\cosh u} - l\cosh u + K]}{(l\cosh u - K)h(l\cosh u)}, \qquad (22)$$

and

$$h(u) = (u + K)e^{-2ud} - u + K\sigma. \qquad (23)$$

¿From (6) we have $h(k) = 0$. The integrands in (20) & (21) have poles at $u = \gamma_1$ and $u = \gamma_2$ where

$$\cosh \gamma_1 = K/l, \qquad \text{and} \qquad \cosh \gamma_2 = k/l. \tag{24}$$

To solve the scattering problem we write the velocity potential in terms of the incident wave and the multipoles as follows

$$\phi = \phi_{\text{inc}} + \sum_{m=0}^{\infty} a^m (\alpha_m \phi_m^a + \beta_m \phi_m^s), \tag{25}$$

where $\alpha_0 = 0$ is included for convenience. To find the unknowns α_m and β_m we apply the body boundary condition $\partial\phi/\partial r = 0$ on $r = a$. For an incident wave of wavenumber K we obtain

$$\alpha_n + \frac{I_n'(la)}{K_n'(la)} \sum_{m=1}^{\infty} \alpha_m a^{m-n} A_{mn}^a = \left(\frac{-1}{a}\right)^n 2i \frac{I_n'(la)}{K_n'(la)} e^{Kf} \sinh n\gamma, \qquad n = 1, 2, \dots \tag{26}$$

$$\beta_n + \frac{I_n'(la)}{K_n'(la)} \sum_{m=0}^{\infty} \beta_m a^{m-n} A_{mn}^s = -\left(\frac{-1}{a}\right)^n \epsilon_n \frac{I_n'(la)}{K_n'(la)} e^{Kf} \cosh n\gamma, \qquad n = 0, 1, \dots, \tag{27}$$

where $\cosh\gamma = 1/\sin\alpha_{\text{inc}}$. Truncation of the systems to $N \times N$ systems is required to find solutions. Convergence is rapid and for the results below a value of $N = 4$ was used. The reflection and transmission coefficients are obtained from the far field form of the velocity potential:

$$T_1 = 1 - \pi e^{Kf} Res[C_L : \gamma_1] \sum_{m=0}^{\infty} (-a)^m [\alpha_m \sinh m\gamma_1 - i\beta_m \cosh m\gamma_1], \tag{28}$$

$$R_1 = \pi e^{Kf} Res[C_L : \gamma_1] \sum_{m=0}^{\infty} (-a)^m [\alpha_m \sinh m\gamma_1 + i\beta_m \cosh m\gamma_1], \tag{29}$$

$$t_1 = \pi e^{kf} Res[C_L : \gamma_2] \sum_{m=0}^{\infty} (-a)^m [-\alpha_m \sinh m\gamma_2 + i\beta_m \cosh m\gamma_2], \tag{30}$$

$$r_1 = \pi e^{kf} Res[C_L : \gamma_2] \sum_{m=0}^{\infty} (-a)^m [\alpha_m \sinh m\gamma_2 + i\beta_m \cosh m\gamma_2]. \tag{31}$$

For an incident wave of wavenumber k we have $l = k \sin\alpha_{\text{inc}}$ and the equations for α_m, β_m are simply (26), (27) with K replaced with k. The equations for R_2 and r_2 are the same as those for R_1 and r_1, but for the transmission coefficients we have

$$T_2 = \pi e^{Kf} Res[C_L : \gamma_1] \sum_{m=0}^{\infty} (-a)^m [-\alpha_m \sinh m\gamma_1 + i\beta_m \cosh m\gamma_1], \tag{32}$$

$$t_2 = 1 - \pi e^{kf} Res[C_L : \gamma_2] \sum_{m=0}^{\infty} (-a)^m [\alpha_m \sinh m\gamma_2 - i\beta_m \cosh m\gamma_2]. \tag{33}$$

Results and discussion

There are many different features of this problem that could be explored. Here we will concentrate on just one, the occurrence of zeros of transmission for particular parameter values. Figure 1 shows the reflection and transmission energies for an incident wave of wavenumber k. The transmission and reflection coefficients for waves of wavenumber K cut in at $Ka \sim 0.313$ which is the critical frequency for this case. For a frequency of $Ka \sim 0.288$ we observe there is zero transmission and full reflection of the waves of wavenumber k. Given that circular cylinders are known to reflect no energy in a single layer fluid of infinite depth this is perhaps a surprising result. Figure 2 shows reflection energies for an incident wave of wavenumber k with varying submergence of the cylinder in the lower layer. The depth of the upper layer, d/a, is fixed and the angle of incidence, α_{inc}, is set greater than the critical angle so that there are no waves of wavenumber K propagating on the free surface. We can see from the figure that there is a value of f/a which gives total reflection for a particular frequency Ka. For cylinders closer to the interface two frequencies at which total reflection occurs exist. A similar effect is observed when fixing the submergence and varying the depth of the upper fluid. The existence of zeros of transmission at certain frequencies raises the possibility that trapped modes may exist in the presence of a pair of circular cylinders submerged in a two layer fluid. This will be the subject of further research. It is also possible to examine scattering by a cylinder in the upper fluid by exactly the same method and results for this case will be presented at the workshop.

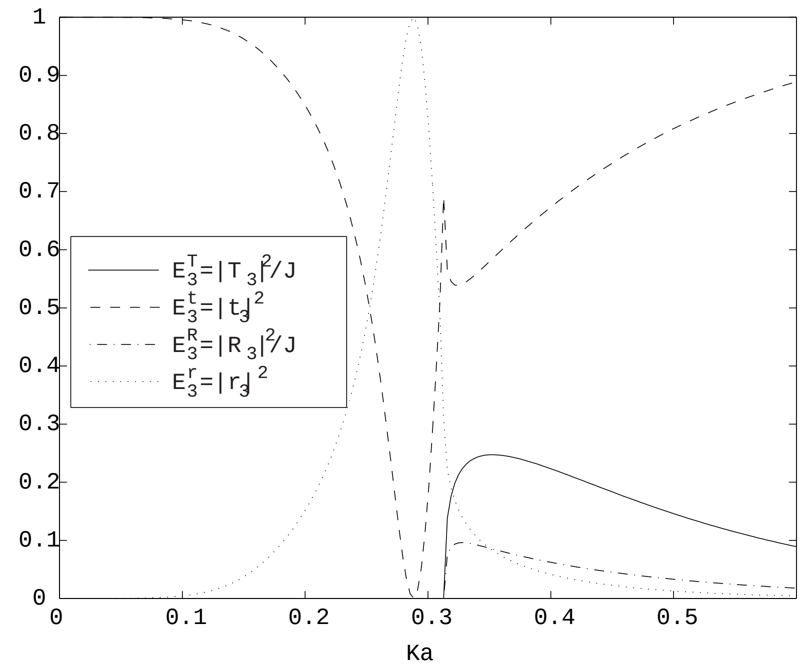

Figure 1: Transmission and reflection energies due to a wave of wavenumber k incident on a cylinder in the lower layer; $\rho = 0.5$, $d/a = 2.0$, $f/a = -1.1$ and $\alpha_{\rm inc} = 0.33$.

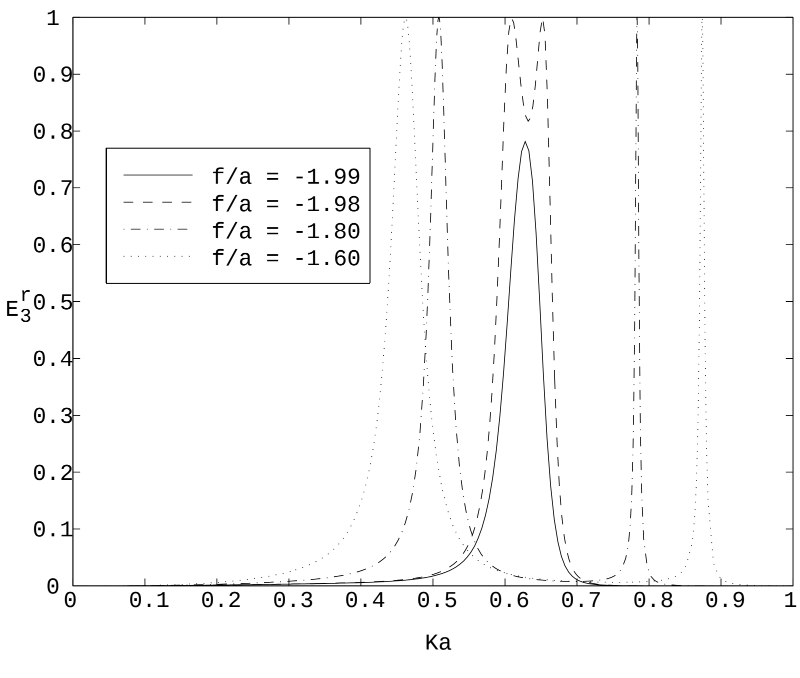

Figure 2: Reflection energies due to a wave of wavenumber k incident on a cylinder in the lower layer; $\rho = 0.5$, $d/a = 2.0$ and $\alpha_{\rm inc} = 0.34$.

References

Linton, C.M. & McIver, M. 1995 'The interaction of waves with horizontal cylinders in two-layer fluids.' *J. Fluid Mechanics*, Vol. 304 , pp. 213–229.