SUPERCRITICAL WAKES IN STRATIFIED FLOWS

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1 Introduction

According to the linear hydrostatic equations of motion, supercritical free-surface flow (Froude number $F > 1$) past a locally confined obstacle gives rise to two oblique wavetrains, oriented at an angle $\cot^{-1}(F^2 - 1)^{1/2}$ to the flow direction, that form a wake downstream, and these theoretical predictions have been confirmed experimentally (see, for example, Baines [1] §2.2). We shall be concerned with analogous phenomena in stratified flow over locally confined topography that have been observed in satellite photographs of atmospheric internal-wave patterns generated by isolated islands (Gjevik & Marthinsen [2]).

In the context of stratified flow, under certain conditions, it is possible to set up a finite-amplitude theory that allows one to examine the role that nonlinear effects in the flow over the topography may play in generating supercritical wakes downstream. More specifically, when the topography is more elongated in the spanwise than in the streamwise direction, the nonlinear three-dimensional (3-D) equations of motion can be handled via a matched-asymptotics procedure. Three specific flow configurations are discussed in detail: (a) nonresonant flow of general (stable) stratification over finite-amplitude topography in a channel of finite depth; (b) resonant, uniformly stratified flow in a channel of finite depth; and (c) vertically unbounded, uniformly stratified flow over finite-amplitude topography. In all three cases, supercritical wakes are found downstream, but in (b) and (c) these wakes are induced by nonlinear interactions precipitated by 3-D effects in the flow over the topography, and are significantly stronger than their linear counterparts.

2 Formulation

We shall be concerned with steady, inviscid, incompressible, stratified flow. It is convenient to work with the streamfunctions Ψ, Φ:

$$\boldsymbol{u} = \nabla\Phi \times \nabla\Psi \ , \tag{1}$$

and take $\rho = \rho(\Psi)$ so that incompressibility and mass conservation are automatically satisfied. Furthermore, assuming that the flow starts from rest, one may show, using the circulation theorem, that the vorticity, $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$, lies in surfaces of constant ρ so that

$$\boldsymbol{\omega} \cdot \nabla\Psi = 0 \ . \tag{2}$$

On the hypothesis of no upstream influence, the flow is undisturbed far upstream ($x \to -\infty$):

$$\Psi \sim U_0 z \ , \quad \Phi \sim y \ , \quad \rho \sim \rho_0(z) \qquad (x \to -\infty) \ ,$$

U_0 and $\rho_0(z)$ being the background flow speed and density profile, respectively. Hence, $\rho = \rho_0(\Psi/U_0)$ and, making use of (2), the momentum equation implies that

$$\boldsymbol{\omega} \cdot \nabla\Phi = \frac{1}{\rho}\frac{d\rho}{d\Psi}\left\{\tfrac{1}{2}U_0^2 - \tfrac{1}{2}\boldsymbol{u}^2 + \frac{g}{U_0}\left(\Psi - U_0 z\right)\right\} \ . \tag{3}$$

Having solved for the density ρ in terms of Ψ, and since $\boldsymbol{\omega} = \nabla \times \nabla\Phi \times \nabla\Psi$, equations (2) and (3) may now be viewed as two equations for determining Ψ and Φ and thereby the velocity field via

(1). This equation set is intractable, however, and, to make further analytical progress, we shall focus on long, weakly 3-D disturbances in a Boussinesq fluid:

$$\mu = \frac{H}{L} \ , \quad \alpha = \frac{L}{D} \ , \quad \beta = \frac{HN_0^2}{g} \to 0 \ ,$$

where N_0 is a characteristic Brunt–Väisälä frequency, H is a vertical lengthscale and L, D are the topography lengthscales in the streamwise (x-) and spanwise (y-) direction, respectively. On the other hand, the nonlinear parameter, $\epsilon = h/H$, h being the topography amplitude, is not assumed to be small.

Making variables dimensionless using L as the characteristic horizontal lengthscale, H as the characteristic vertical lengthscale and U_0 as the velocity scale, and after the re-scaling

$$\Psi = z + \psi(x, Y, z) \ , \qquad \Phi = y + \alpha\phi(x, Y, z) \ ,$$

the governing equations (2) and (3) become

$$\Psi_{zz} + \frac{N^2(\Psi)}{F^2}\,(\Psi - z) = O(\alpha^2, \mu^2) \ , \tag{4}$$

$$J(v, \Psi) = \Psi_z\, \Psi_{zY} - \Psi_Y\, \Psi_{zz} + O(\alpha^2, \mu^2) \ , \tag{5}$$

where $F = U_0/(N_0 H)$ is the Froude number, $Y = \alpha y$ is a stretched spanwise coordinate, and

$$v = J(\Psi, \phi)$$

is the spanwise velocity component, $J(a, b) = a_x b_z - a_z b_x$ being the Jacobian.

Based on (4) and (5), one may devise a perturbation solution scheme: first, the leading-order approximation to Ψ, $\Psi^{(0)} = z + \psi^{(0)}$ say, can be readily obtained by solving (4) subject to appropriate boundary conditions on the topography and the upper boundary of the flow. The spanwise velocity component, $v^{(0)}$, then follows from (5) by integrating along x on surfaces of constant $\Psi^{(0)}$:

$$v^{(0)} = \frac{\partial}{\partial Y} \int^x \mathrm{d}x'\, \psi_z^{(0)} \Big|_{\Psi^{(0)}} \ .$$

This perturbation expansion becomes nonuniform, however, far upstream and downstream of the topography. It turns out that the far-field disturbance is weakly nonlinear but not weakly 3-D, invalidating the scalings chosen earlier, and it is governed by the linear 3-D hydrostatic equations to leading order. To fully determine the flow field, it is necessary to match the nonlinear, weakly 3-D disturbance over the topography with the weakly nonlinear, 3-D far-field response.

3 Summary of Results

The matching procedure indicated above was carried out for three particular flow configurations. Specifically,

(a) Non-resonant flow over finite-amplitude topography in channel of finite depth. In this instance, there is an infinity of long-wave modes and it is assumed that F is not close to any of the critical Froude numbers:

$$\cdots < F_M < F < F_{M-1} < F_{M-2} < \cdots < F_1 \ .$$

The downstream response consists of an infinite number ($n \geq M$) oblique wavetrains, each oriented at angle $\tan^{-1}(F_n/\sqrt{F^2 - F_n^2})$ to the flow direction, forming multiple supercritical wakes. The amplitudes of these wakes are not affected seriously by nonlinearity in the near-field flow. On the other hand, 3-D effects inhibit breaking of the flow over the topography, increasing the critical steepness for overturning by $O(\alpha)$.

(*b*) Uniformly stratified, resonant flow in a channel of finite depth. When the flow speed is close to the speed of one of the long-wave modes,

$$F \approx F_M \ ,$$

the flow is resonant. Furthermore, when the flow is uniformly stratified, small-amplitude topography induces a finite-amplitude response under resonance conditions [3]. Qualitatively, the geometry of the wakes generated downstream is similar to those found in the non-resonant case. The generation mechanism, however, is quite different here: nonlinear interactions in the flow over the topography play an important part, and the amplitudes of the downstream wakes are much stronger than their linear counterparts (by a factor of $O(\alpha^{-2})$).

(*c*) Vertically unbounded, uniformly stratified flow. Here the flow may be regarded as resonant for all flow speeds because the spectrum of long-wave modes is continuous. Furthermore, according to linear theory, the 3-D wave pattern induced by locally confined topography remains locally confined. We find that, owing to the same nonlinear mechanism as in (*b*) above, multiple wakes are generated downstream so, in contrast to linear theory, the nonlinear wave pattern is not locally confined. The wake angles are given by $\tan^{-1}(1/\sqrt{n^2-1})$ $(n \geq 2)$.

References

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[3] Grimshaw, R. H. J. and Yi, Z. Resonant generation of finite-amplitude waves by the flow of a uniformly stratified fluid over topography. *J. Fluid Mech.*, **229**, 603–628, 1991.