A Fourier-Boussinesq method for nonlinear wave propagation on a variable depth fluid

Y. Agnon* and H. B. Bingham†

This abstract describes a new Boussinesq-type model which overcomes the restriction to small values of kh which is inherent in traditional Boussinesq methods ($k = |\vec{k}|$ the wavenumber and h the water depth.) The term Boussinesq-type model is used here in a broad sense to describe a two-dimensional approximation to the exact potential flow problem, where the vertical coordinate has been eliminated from the formulation. This elimination is done, without approximation, by representing the solution as an infinite power series in z. The way in which this series is truncated (as well as the particular choice of velocity variable) determines the final form of the Boussinesq-type model. Traditional Boussinesq methods obain approximate linear dispersion relations (expressed in terms of non-dimensional wave celerity $\frac{c^2}{gh} = \frac{\omega^2}{k^2 gh} = \frac{\tanh kh}{kh}$) which are rational functions of $(kh)^2$. The exact relation is however trancendental, and tends to $\frac{1}{kh}$ as $kh \to \infty$, which can not be matched asymptotically by a function of $(kh)^2$. By introducing the FFT, and thus allowing a fast evaluation of the Hilbert transform, algebraic approximations to the dispersion relation which are asymptotically correct in both limits of kh can be investigated.

Following [5] (and others) we consider weakly nonlinear solutions to the exact Laplace problem by expressing the free-surface boundary conditions in terms of the potential at the free-surface,

$$\begin{aligned} \eta_t &= \quad (1 + \nabla\eta \cdot \nabla\eta)\tilde{\phi}_z - \nabla\tilde{\phi} \cdot \nabla\eta \\ \tilde{\phi}_t &= \quad -g\eta - \tfrac{1}{2}(\nabla\tilde{\phi})^2 - \tfrac{1}{2}(1 + \nabla\eta \cdot \nabla\eta)\tilde{\phi}_z^2 \end{aligned} \tag{1}$$

where $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ and $\tilde{\phi} = \phi(\vec{x}, \eta, t)$. The surface potential is then expressed as a perturbation expansion in the wave slope ϵ, and each perturbation potential is Taylor expanded from $z = 0$ to $z = \eta$. Thus,

$$\tilde{\phi} = \sum_{m=1}^{M} \sum_{k=0}^{M-m} \frac{\eta^k}{k!} \frac{\partial}{\partial z} \hat{\phi}^{(m)} \tag{2}$$

where $\hat{\phi} = \phi(\vec{x}, 0, t)$. Expressed in this form, the closure which allows us to step forward in time is a relation for the vertical component of velocity w, in terms of the horizontal component of velocity $\vec{u}$, where both quantities are evaluated on $z = 0$. This relationship must naturally satisfy the Laplace equation and the bottom boundary condition

$$\phi_z + \nabla\phi \cdot \nabla h = 0 \quad z = -h, \tag{3}$$

in which case the problem is solved. To be attractive computationally, the method should be of order N the number of unknowns on the free-surface.

Consider first a flat bottom. If the solution is expanded in a Taylor series about z=0, and the Laplace equation is invoked then

$$\phi(\vec{x}, z, t) = \sum_{n=0}^{\infty} (-1)^n \left(\frac{z^{2n}}{2n!} \nabla^{2n} \hat{\phi} + \frac{z^{2n+1}}{(2n+1)!} \nabla^{2n} w \right). \tag{4}$$

The bottom boundary condition may now be expressed as

$$\mathrm{Cos}(h\,\nabla)\, w + \mathrm{Sin}(h\,\nabla)\, \vec{u} = 0 \tag{5}$$

*Tecnion, Haifa, Israel, agnon@cameri2.technion.ac.il
†Danish Technical U., Lyngby, Denmark, hbb@imm.dtu.dk

where the Sin and Cos Taylor series operators are defined by

$$\mathrm{Sin}(L\,\nabla) = L\,\nabla - \frac{L^3\,\nabla^3}{6} + \frac{L^5\,\nabla^5}{120} + \ldots$$
$$\mathrm{Cos}(L\,\nabla) = 1 - \frac{L^2\,\nabla^2}{2} + \frac{L^4\,\nabla^4}{24} + \ldots \tag{6}$$

and ∇ is understood as the gradient when applied to a scalar and the divergence when applied to a vector. If the operators in Equation (6) are truncated, and replaced by finite-difference operators, the result is a sparse matrix equation for w in terms of $\vec{u}$. This corresponds to a classical Boussinesq method. Imbedded in Equation (5) is the linear dispersion relation. To see this note that in the frequency domain the operator $\nabla = ik$ (in one horizontal dimension), so we can write $W = -\mathrm{Tan}(h\nabla)U = -i\mathrm{Tanh}(kh)\vec{U}$, where U, W are the frequency-domain velocity components and the Tan and Tanh operators are defined in the way of Equation (6). The linear free-surface boundary condition says that $g\nabla w = -\vec{u}_{tt}$ or $W = \frac{\omega^2}{ig\vec{k}}U$. Combining these two expressions gives the exact linear dispersion relation $\omega^2 = gk\tanh{(kh)}$. Thus a classical Boussinesq method can be thought of coming from the equation

$$W = \frac{\mathrm{Tanh}(\vec{k}h)}{\vec{k}h} ih\vec{k}\cdot\vec{U}, \tag{7}$$

where $\mathrm{Tanh}(\vec{k}h)/\vec{k}h$ is approximated by truncated Taylor series expansion of $\mathrm{Sinh}\vec{k}h$ and $\mathrm{Cosh}\vec{k}h$, and $ih\vec{k}$ is associated with $h\nabla$ to get a method in physical space. Such classical methods unfortunately diverge beyond $kh = \pi/2$ due to the first singularity of tanh on the imaginary axis. This problem can however be avoided by replacing the Taylor expansions by Padé expansions, resulting in modified (or enhanced) Boussinesq methods. For example, the Padé(4,4) expansion of $\mathrm{Tanh}(\vec{k}h)/\vec{k}h$ is

$$\frac{1 + \frac{1}{9}\kappa^2 + \frac{1}{945}\kappa^4}{1 + \frac{4}{9}\kappa^2 + \frac{1}{63}\kappa^4}$$

where $\kappa = \vec{k}h$, which leads to a very accurate method out to $kh = 6$ (see [3].)

Here we go one step further by introducing the Hilbert transform operator H, which is most conveniently written in Fourier space as $\mathcal{F}\{H\phi\} = -i\mathcal{F}\{\phi\}$ where $\mathcal{F}$ indicates a Fourier transform. (The operator applies a 90° phase shift in Fourier space.) This allows even powers of κ to appear in the rational approximation for $\tanh\kappa/\kappa$. The trick is to use the identity

$$\tanh{(\kappa)} \equiv \frac{p_1}{1+p_1}, \qquad p_1 = \frac{\sinh{(\kappa)}}{\cosh(\kappa) - \sinh(\kappa)}. \tag{8}$$

By taking a Padé(m,n) expansion of p_1 with $n < m$ the correct assymptotics are obtained for both limits of κ. Figure 1 shows the relative error in linear dispersion for a Padé(4,1) and a Padé(8,2) approximation of p_1. The specific methods obtained in these two cases are:

$$(15 + 9\kappa + 9\kappa^2 + 4\kappa^3 + \kappa^4)W = -(15 + 9\kappa + 4\kappa^2 + \kappa^3)ih\vec{k}\cdot\vec{U} \tag{9}$$

and

$$\begin{aligned}(14175 + 8505\kappa + 9135\kappa^2 + 4410\kappa^3 + 1575\kappa^4 + 420\kappa^5 + 84\kappa^6 + 12\kappa^7 + \kappa^8)W \\ = -(14175 + 8505\kappa + 4410\kappa^2 + 1575\kappa^3 + 420\kappa^4 + 84\kappa^5 + 12\kappa^6 + \kappa^7)ih\vec{k}\cdot\vec{U}\end{aligned} \tag{10}$$

respectively, where the corresponding differential equation is obtained by associating $hi\vec{k}$ with $h\nabla$ and $-i$ with H. The agreement with the exact linear dispersion relation for all wavenumbers is remarkable.

In order to create a method which is easily implemented, it is desirable perform some further manipulation. Since the left hand sides of Equations (9) and (10) represent the implicit part of the solution, it is convenient to convert these polynomials into ones with only even powers of kh. (When odd powers are retained, the solution must be obtained iteratively.) It turns out that any polynomial $f(x)$ can be written with only even powers by multiplying it with $f(-x)$. Multiplying both sides of the equation by this polynomial produces an explicit matrix equation for w in terms of $\vec{u}$, at the cost of doubling the highest order of the derivative which must be calculated. Note also that if $\kappa = \kappa_1$ is a root of the original polynomial, then both κ_1 and $-\kappa_1$ are roots of the new polynomial; so to get a stable method, the original polynomial must have no real roots. The odd powers of κ which remain in the numerator are evaluated by taking horizontal derivatives of $\nabla H \vec{u}$ which is found using the FFT.

Our first application of the method is to the infinite depth, non linear standing wave solution of [4]. Figure 2 shows the wave elevation at the center of the tank as a function of time, calculated using the Padé(8,2) method. 65 points have been used on the free-surface and 40 points per wave period, while the non linearity retained in the free-surface boundary condition corresponds to $M = 5$ in Equation (2). After 25 periods the mass has been conserved to within 2x10^{-7} and energy to within 4x10^{-5} of the initial conditions. This is a periodic solution, and therefore well suited to a method which incorperates the FFT. In order to treat non-periodic problems, we apply the method presented in [2]. The idea is to split the solution into two potentials, one of which is known analytically (or solved by another method), while the other is found by the above described technique. The known potential corresponds to a flux into or out of the domain and can represent a wave maker or absorber for example. This solution can be any appropriate solution to the Laplace equation and the bottom boundary condition, as long as the sum of the two potentials satisfies the the free-surface boundary conditions.

When the bottom slope is retained in the bottom boundary condition, it takes the form

$$\mathrm{Cos}(h\,\nabla)\,w_0 + \mathrm{Sin}(h\,\nabla)\cdot\vec{u}_0 + \nabla\,h\cdot[\mathrm{Cos}(h\,\nabla)\cdot\vec{u}_0 - \mathrm{Sin}(h\,\nabla)\,w_0] = 0. \tag{11}$$

In this case, derivation of the approximate equation is more involved. When multiplying powers of $h\nabla$, terms involving ∇h, $(\nabla h)^2$, $\nabla^2 h$ etc. arise. Since h is variable, ∇ and h do not commute, and their ordering affects the product. The details will appear in a future publication [1]. We present however some prelimary calculations. Figure 3 shows a wave with period $T = 8$s shoaling up a sloping beach as computed by the Padé(8,2) method using the linear free-surface boundary conditions. Also shown is the amplitude envelope predicted by linear theory, and the depth variation.

References

[1] Agnon, Y. and Bingham, H. B. A Boussinesq type model for fully dispersive water waves. *In preparation.*

[2] Agnon, Y. and Bingham, H. B. A non-periodic spectral method with appliction to nonlinear water waves. in *European J. Mech. B/Fluids* **18:3** 527–534, 1999.

[3] Agnon, Y., Madsen, P. A. and Schäffer, H. A. A new approach to high order Boussinesq models. in *Journal of Fluid Mechanics* **399** 319–333. 1999

[4] Agnon, Y. and Glozman, M. Periodic solutions for a complex Hamiltonian system: New standing water waves. in *Wave Motion* **24** 139–150, 1996.

[5] Dommermuth, D. and Yue, D. K. P. A high-order spectral method for the study of nonlinear gravity waves. in *J. Fluid Mech.* **184** 267–288, 1987.

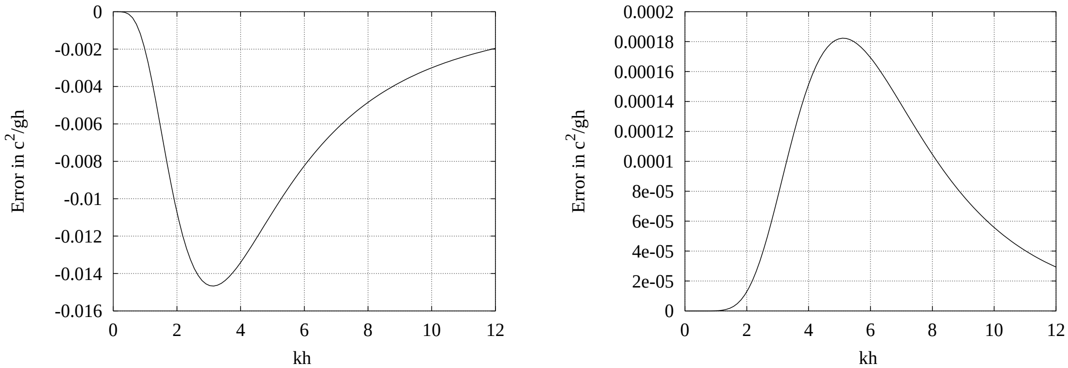

Figure 1: Relative errors in dispersion for Fourier-Boussinesq Padé(4,1) and Padé(8,2) methods.

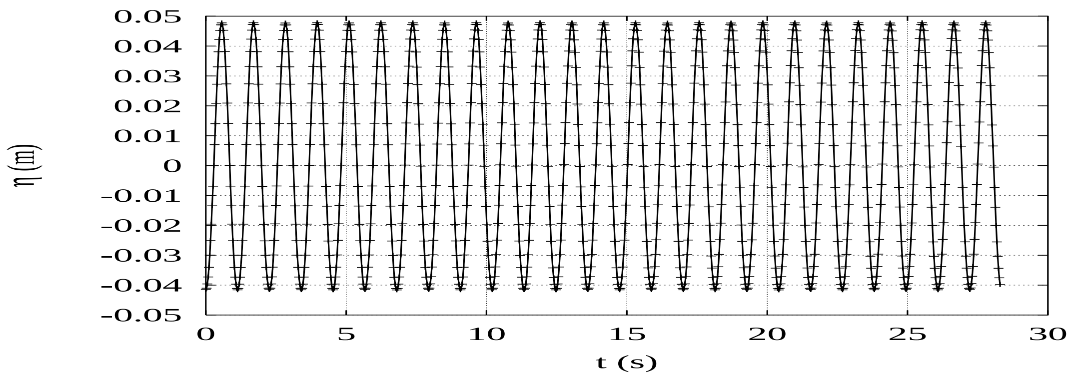

Figure 2: Time history of the elevation at the center of the tank with a non linear standing wave as the initial condition.

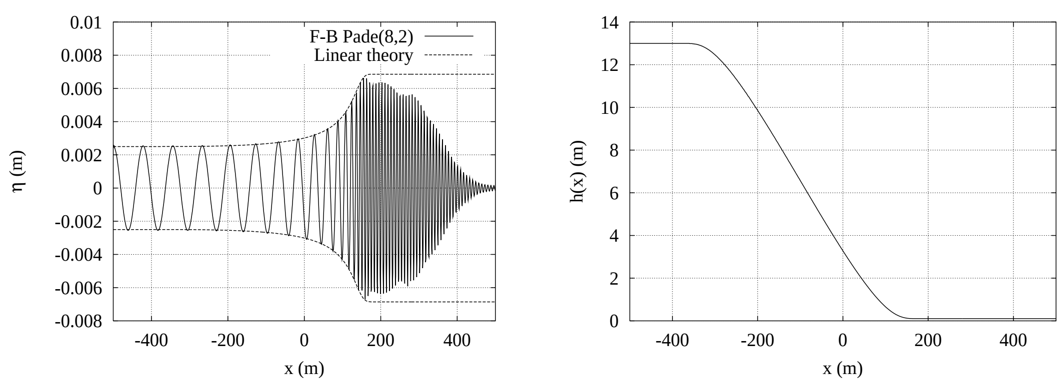

Figure 3: Linear shoaling for a wave of period $T = 8$s over the bottom shown. A wavemaker generates the waves from the left boundary and a sponge layer absorbs them at the right boundary.