Nonlinear gravity-capillary waves generated by a moving disturbance

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Periodic gravity-capillary waves have been studied since the beginning of the century. The evidence of multiple solutions was first shown by Wilton [A] who included surface tension in Stokes classical expansion for pure gravity waves. This work was later extended by Pierson and Fife [B] and others. Fully nonlinear solutions were obtained by Schwartz and Vanden-Broeck [C] and Chen and Saffman [D]. More recently new results were discovered for gravity-capillary solitary waves. In particular, it was found that there are generalized solitary waves (i.e. "solitary" waves with tails of ripples of constant amplitude in the far field) and solitary waves with decaying oscillatory tails in the far field. These waves were first calculated numerically for the full Euler equations by Hunter and Vanden-Broeck [E] and Vanden-Broeck and Dias [F], respectively.

In spite of the progress on periodic and solitary waves, there are still many open questions for gravity-capillary flows generated by a moving disturbance (floating object, submerged object or pressure distribution). Rayleigh derived a linearized solution for a small distribution of pressure moving at a constant velocity U at the surface of a fluid of infinite depth. His results indicate that there is a critical value U_c, such that there are no waves on the free surface when $U < U_c$. On the other hand, for $U > U_c$, there are two trains of waves. One train is dominated by gravity and the other by surface tension. A unique solution is obtained by imposing the energy radiation condition which requires the train of waves dominated by gravity to be behind the disturbance and the one dominated by surface tension to be at the front. Rayleigh's solution is accurate for $U \neq U_c$ in the limit as the magnitude of the pressure distribution approaches zero. However it is nonuniform as $U \to U_c$: for a given distribution of pressure, the displacement of the free surface becomes unbounded as $U \to U_c$. Vanden-Broeck and Dias re-examined the problem for $U < U_c$. Their results show that this nonumiformity is associated with the existence of are additional branches of solutions which can be viewed as perturbations of solitary waves. So far no corresponding results have been obtained for $U > U_c$.

We will present new numerical and analytical results for $U > U_c$. The problem is more complicated than the one considered by Vanden-Broeck and Dias [F] because there are train of waves in the far field instead of a flat free surface. In particular numerical calculations require the derivation of appropriate boundary conditions to truncate the domain in the far field. We combine several numerical procedures, we used successfully in previous work. These include series truncation techniques and boundary integral equation methods. The idea of the series truncation method is to identify a rapidly convergent series representation for the solution which satisfies all the appropriate partial differential equations (Laplace equation for potential flows) and all the linear boundary conditions. This often requires local analysis to identify and remove singularities associated with corners, stagnation points, etc. The series is then truncated after a finite number of terms and the unknown coefficients are found by satisfying the nonlinear boundary conditions (the pressure condition for free surface flows) at chosen collocation points. We found that this method is highly accurate for both periodic and solitary waves. The boundary integral equation methods are based on a reformulation of the problem as a system of nonlinear integro-differential equations for the unknown quantities on the free surface. These equations are then discretized and the resultant algebraic equations are solved by Newton's method. Experience has shown that the details of the discretization depends on the problems to be solved and we have achieved success for many different types of free surface flows such as generalized solitary waves and axisymmetric waves.

An important ingredient of the numerical scheme, is the appropriate imposition of the radiation condition. In his linear theory, Rayleigh used an "artificial" viscosity. In our nonlinear calculations, we model the "real" viscosity by using the quasi-potential approximation (see for example Ruvinsky, Feldstein and Freiman [G]). We present results for a moving disturbance on the free surface. The computations show that a turning point is ultimately reached as $U \to U_c$. This implies the surprising result that the solutions for $U > U_c$ are not connected to those for $U < U_c$.

We will also present results with surfactants. This is an attempt to develop a complete theory involving the effect of viscosity and variable surface tension. Preliminary results have been obtained using the Boussinesq-Scriven constitutive relationship (see [H]) and work is presently been completed to check the consistency of the various approximations.

Bibliography

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