Non-linear effects in trapped modes in the case of gravitational waves in a channel with an elastic plate

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Abstract

It is shown that in the thin infinite plain channel filled by inviscid incompressible fluid with an elastic plate on the bottom there are both moving waves and standing waves localized in the region of the plate. Non-linear effects due to the free surface of the fluid are studied. Using rigorous perturbation theory the dependencies of all main charactistics of the system on the amplitude ε of plate oscillations are found as a power series of ε. It is shown that the amplitude of moving waves may be also presented as a power series of ε. We can vanish the first term of this series by the choosing of plate width. In this case the amplitude of standing waves will be proportional to ε^0, whereas the amplitude of moving waves will be proportional to ε^2.

Starting from the pioneering paper by F. Ursell [1] where the existence of trapped modes was first demonstrated, this phenomenon is of great attention in many systems. But the most of theoretical considerations concern only with the linear approximation [2]–[5], therefore, non-linear effects of trapped modes is not clear understood. In order to investigate the role of non-linear effects we consider the simplest case that is the thin plane channel filled by inviscid, incompressible fluid with an elastic plate on the bottom. Non-linearity arises from the free surface of the fluid.

It is shown that in this case the dimensionless wave profile $u(z,\tau)$ is described by the Boussinesq equation with additional term corresponding to the oscillating plate

$$u_{zz} - u_{\tau\tau} - \omega^2 \sin\omega\tau \cdot \theta(z) + \frac{3}{2}\varepsilon(u^2)_{zz} + \frac{\delta}{3}u_{zz\tau\tau} = 0, \qquad (1)$$

where z and τ are dimensionless coordinate and time, $\theta(z)$ is the theta-function

$$\theta(x) \equiv \begin{cases} 1, & |x| \leqslant 1 \\ 0, & |x| > 1 \end{cases} \tag{2}$$

(elastic plates begins at the point $z = -1$ and finishes at the point $z = 1$), ε is the small parameter that equals to the amplitude of plate oscillations ξ_0 divided by channel depth H, δ is the squared division of channel depth H by plate size a (δ is assumed to be small enough). In the linear approximation when $\varepsilon = 0$ moving waves are of the amplitude $\sin(\omega_0/\sqrt{1-\delta\omega_0^2/3})$. Therefore, moving waves do not exist only in the case of $\sin(\omega_0/\sqrt{1-\delta\omega_0^2/3}) = 0$ (trapped modes) or

$$\omega_0 = \omega_k \equiv \frac{k\pi}{\sqrt{1+\frac{\delta}{3}(k\pi)^2}}, \tag{3}$$

($k = 1, 2, 3, \ldots$). But the non-linear term disturb this result. In the framework of rigorous perturbation theory [6] it is shown that the trapped modes in zeroth approximation exist as well, but the trapped frequencies and required mass M of the plate should depend from the amplitude of oscillations ξ_0. Moreover, in the first-order approximation the moving waves with the amplitude $\frac{\sin\left(\delta(k\pi)^3\right)}{\delta(k\pi)^2}$ appear. In principle, we may equate this term to zero by the choose of δ and provide the trapping modes in the first approximation too. Finally, in the first-order approximation (already in dimensional terms) we have for $\omega = \omega_k$

$$\omega_k = k\pi\frac{\sqrt{gH}}{a}\,\frac{1+\left(\dfrac{3\xi_0 a}{4k\pi H^2}\right)^2}{\sqrt{1+\dfrac{1}{3}\left(\dfrac{k\pi H}{a}\right)^2}}. \tag{4}$$

$$u(x,t) = \xi_0 \left[1 + (-1)^{k-1} \cos \frac{k\pi x}{a}\right] \theta\left(\frac{x}{a}\right) \sin \omega_k t +$$
$$\frac{(-1)^{k-1}}{2} \xi_0 \sin\left(k\pi \frac{|x|}{a} - \omega_k t\right) \theta\left(\frac{|x|}{a} - \frac{\omega_k}{k\pi} t\right) +$$
$$\frac{3\xi_0^2 a^2}{8(k\pi)^2 H^3} \sin\left(\frac{H^2}{a^2}(k\pi)^3\right) \sin\left[2\left(k\pi\frac{|x|}{a} - \omega_k t\right)\right] \left[\theta\left(\frac{|x|}{a}\right) - 1\right], \quad t > \frac{2a}{\sqrt{gH}} \tag{5}$$

In the case when the amplitude of moving waves equals to zero in the first-order approximation we have for the plate mass $M = M_k$, where

$$M_k = \frac{a}{(k\pi)^2} \frac{\varkappa}{gH} \left[1 + \left(\frac{3\xi_0 a}{4k\pi H^2}\right)^2\right]^{-2}, \tag{6}$$

g is the gravity acceleration, $\varkappa$ is the elastic rigidity. Thus, although the non-linear effects disturb the trapping modes, choosing the geometry of channel and plate we may provide the effect of trapping modes in the first-order approximation in terms of ϵ (the amplitude of the moving waves will be proportional to ϵ^2).

References

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