ON THE VALIDITY OF MULTIPOLE EXPANSIONS

by

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1 Introduction

The Method of Multipoles is an effective method for solving certain scattering problems in linear wave theory, particularly those involving immersed and submerged circles (in two dimensions) and spheres (in three dimensions). An example is the submerged sphere between parallel walls which has been treated by G.X.Wu and for which an alternative treatment was suggested by me at the last Workshop. During the discussion David Evans raised the following question : Can the potential always be expressed as the sum of the appropriate multipoles ? For the proof we need to find good bounds for the image potentials and there is no simple method for this. In the present note I shall show that there is a simple argument for two dimensions, and a more complicated argument for three dimensions. I have no serious doubts about the validity of multipole expansions, (including the expansions in Wu's problem,) but it is curious that the mathematical arguments are not more obvious.

2 The circle

We consider first the classical problem of the submerged circle in two dimensions. The velocity potential $\phi(x,y)e^{-i\omega t}$ is defined in the part of the region $(-\infty < x < \infty,\ 0 < y < \infty)$ outside the circle

$$x^2 + (y-f)^2 = a^2,$$

where $a < f$, and satisfies Laplace's equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi(x,y) = 0. \qquad (2.1)$$

The free-surface condition is

$$K\phi + \frac{\partial \phi}{\partial y} = 0 \text{ on } y = 0, \qquad (2.2)$$

where $K = \omega^2/g$. On the circle the normal velocity is prescribed,

$$\frac{\partial \phi}{\partial r} = U(\theta). \qquad (2.3)$$

where $x = r\sin\theta$, $y = f + r\cos\theta$. Actually this boundary condition is not used in the following argument. There is also a radiation condition: at $x = \pm\infty$ the waves travel outwards.

Clearly $U(\theta)$ is the sum of an even and an odd function of θ. We shall assume that $U(\theta)$ is an even function, an analogous theory can evidently be given for odd functions. For the sake of simplicity we shall also assume that $\int_{-\pi}^{\pi} U(\theta)\, d\theta = 0$. (When this last condition is not satisfied a wave source

$$G_0(x,y) = \frac{1}{2}\log\frac{x^2 + (y-f)^2}{x^2 + (y+f)^2} - 2\int_0^\infty e^{-k(y+f)}\cos kx\,\frac{dk}{k-K} \qquad (2.4)$$

must be added.) There is one obvious method: According to Green's theorem, the potential can be expressed as a distribution of wave sources and wave dipoles over the submerged circle (or sphere) . It is therefore sufficient to show that wave sources or dipoles can be expressed as a series of multipoles at the centre of the circle. This construction has been carried out for a half-immersed circle in [Ursell 1981] but the argument is elaborate. Here a much simpler argument will be given. We assume that a solution

$\phi(x, y)$ exists, and we wish to show that this potential can be expressed as the sum of multipoles at the centre of the circle, which (as is well known, see [Ursell 1950],) are of the form

$$G_m(x,y) = \frac{\cos m\theta}{r^m} + \frac{(-1)^m}{(m-1)!}\int_0^\infty k^{m-1}\,\frac{k+K}{k-K}\,e^{-k(y+f)}\cos kx\;dk, m = 1,2,3,\cdots \qquad (2.5)$$

where the contour of integration passes below $k = K$ to satisfy the radiation condition. Note that the integrand in (2.5) is a solution of Laplace's equation.

PROOF: It is well known that. if the potential exists, it can be expanded in the annulus $a < r < f$ as a Laurent series of the form

$$\phi(x,y) = \sum_1^\infty \cos m\theta\left(p_m\frac{a^m}{r^m} + q_m\frac{r^m}{f^m}\right) \qquad (2.6)$$

where the series

$$\sum_1^\infty \cos m\theta\; p_m\frac{a^m}{r^m}$$

converges when $a < r < \infty$, and the series

$$\sum_1^\infty \cos m\theta\; q_m\frac{r^m}{f^m}$$

converges when $0 < r < f$, actually when $0 < r < 2a - f$. In particular, we have a bound

$$|p_m| < M(a')(a'/a)^m$$

for any $a' > a$. Now consider the expression

$$\Phi(x,y) = \sum_1^\infty p_m a^m G_m(x,y), \qquad (2.7)$$

where the coefficients p_m are the same as in (2.6) We shall show that this expression is a uniformly convergent series and thus defines a potential everywhere in the field of flow . For this purpose we find bounds for the image potentials

$$\frac{(-1)^m}{(m-1)!}\int_0^\infty k^{m-1}\,\frac{k+K}{k-K}\,e^{-k(y+f)}\cos kx\;dk, \qquad (2.8)$$

where, as before, the contour of integration passes below the pole $k = K$. (It is this pole in the integrand which complicates the mathematical argument.) In (2.8) we write

$$\cos kx = \frac{1}{2}e^{ik|x|} + \frac{1}{2}e^{-ik|x|}. \qquad (2.9)$$

Then

$$\int_0^\infty k^{m-1}\frac{k+K}{k-K}e^{-k(y+f)}e^{ik|x|}dk = \int_0^{\infty\exp(i\eta)} k^{m-1}\frac{k+K}{k-K}e^{-k(y+f)}e^{ik|x|}dk \qquad (2.10)$$

$$+\; 4\pi i K^m e^{-K(y+f)}e^{iK|x|} \qquad (2.11)$$

$$= I(m,+) + 4\pi i K^m\; e^{-K(y+f)}e^{iK|x|},\ \text{say}, \qquad (2.12)$$

where the term (2.11) is the residue at the pole $k = K$ and where the acute angle η will be defined later: see(2.17) below. Note that the integrands in (2.10) and the term (2.11) are solutions of Laplace's equation. Similarly

$$\int_0^\infty k^{m-1}\,\frac{k+K}{k-K}\,e^{-k(y+f)}e^{-ik|x|}\,dk = \int_0^{\infty\exp(-i\eta)} k^{m-1}\,\frac{k+K}{k-K}\,e^{-k(y+f)}e^{-ik|x|}\,dk \qquad (2.13)$$

$$= I(m,-),\ \text{say}. \qquad (2.14)$$

In (2.10) we write $k = \sigma e^{i\eta}$ and note that $|\exp(ik|x|)| \leq |\exp(-i\sigma|x|\cos\eta)| \leq 1$, and that

$$\left|\frac{\sigma e^{i\eta} + K}{\sigma e^{i\eta} - K}\right| \leq \cot(\eta/2).$$

Then

$$|I(m,+)| \leq \int_0^\infty \sigma^{m-1} \cot(\eta/2) \exp\{-\sigma(y+f)\cos\eta\}\, d\sigma = \frac{(m-1)!\cot(\eta/2)}{(y+f)^m \cos^m \eta}, \tag{2.15}$$

with the same bound for $|I(m,-)|$. Thus the contribution of this part of the image potential to the series (2.7) is bounded by

$$\frac{1}{2}\sum_{m=1}^{\infty} \frac{|p_m| a^m}{(m-1)!}(|I(m,+)| + |I(m,-)|) < M(a')\sum_{m=1}^{\infty} \frac{(a')^m \cot(\eta/2)}{(y+f)^m \cos^m \eta}, \tag{2.16}$$

and this series converges uniformly for all $y \geq 0$, provided that

$$\cos\eta \geq a'/f, \tag{2.17}$$

i.e. provided that the angle η is small enough. Now consider the contribution to $\Phi(x,y)$ from the terms (2.11) . This is bounded by the series

$$\sum_{m=1}^{\infty} |p_m|\, a^m .2\pi. \frac{1}{(m-1)!} K^m.e^{-K(y+f)} < 2\pi M(a') e^{-K(y+f)} \sum_{m=1}^{\infty} \frac{(Ka')^m}{(m-1)!}, \tag{2.18}$$

a convergent series. Thus (2.7) defines a potential in the whole field of flow.

Consider now the difference potential

$$\phi(x,y) - \Phi(x,y)$$

which is defined in the whole field of flow . In the annulus $a < r < f$ the Laurent expansion contains no negative powers (since the coefficients p_m in (2.6) and (2.7) are identical), and $\phi - \Phi$ is thus defined in the whole of the half-plane $(-\infty < x < \infty,\ 0 < y < \infty)$, including the interior of the circle. By a well-known uniqueness theorem it follows that

$$\phi - \Phi = A.e^{-Ky}\cos Kx,$$

and this satisfies the radiation condition only if $A = 0$. This completes the proof of the expansion theorem.

We have now shown that any solution of our boundary value problem must have the form (2.7). To show that a solution actually exists we expand the terms in (2.7) in polar coordinates and apply the boundary condition (2.3). An infinite system of equations is obtained for the unknowns p_m see e.g. [Ursell 1950].

3 The sphere

We may now attempt the same method for the submerged sphere, but this leads to unexpected difficulties. Only a brief outline can be given. Let the velocity potential be denoted by $\phi(x,y,z)$, and let us assume, for the sake of simplicity, that ϕ is an even function of z. We write $x = r\sin\theta\cos\alpha$, $y = f + r\cos\theta$, $z = r\sin\theta\sin\alpha$. The boundary condition (2.3) is replaced by

$$\frac{\partial\phi}{\partial r} = U(\theta,\alpha) = \frac{1}{2}U_0(\theta) + \sum_{m=1}^{\infty} U_m(\theta)\cos m\alpha. \tag{3.1}$$

Then the typical multipole potential can be shown to be

$$G_n^m = \frac{P_n^m(\cos\theta)}{r^{n+1}}\cos m\alpha + \frac{(-1)^n}{(n-m)!}\int_0^\infty \frac{k+K}{k-K} k^n e^{-k(y+f)} J_m(k\rho)\, dk\ \cos m\alpha. \tag{3.2}$$

We must find a bound for the image potential

$$\frac{(-1)^n}{(n-s)!}\int_0^\infty \frac{k+K}{k-K}\, k^n e^{-k(y+f)} J_s(k\rho)\, dk\, \cos s\alpha, \tag{3.3}$$

which appears in (3.2). The obvious analogue to (2.9) is the decomposition

$$J_s(k\rho) = \frac{1}{2}\left(H_s^{(1)}(k\rho) + H_s^{(2)}(k\rho)\right), \tag{3.4}$$

but for $s \geq 1$ this evidently leads to integrals which are divergent at $k = 0$, and the earlier method is no longer applicable. Instead, in the upper-half k-plane we use the function $\chi_s^{(1)}(Z)$ defined by

$$\chi_s^{(1)}(Z) = \frac{1}{\pi}\int_0^\pi \exp(iZ\sin v - isv)\, dv = J_s(Z) - i\mathbf{E}_s(Z), \tag{3.5}$$

where $\mathbf{E}_s(Z)$ is H.F.Weber's function ([Watson 1922], ch.10). Then, when $Y \geq 0$,

$$|\chi_s^{(1)}(X+iY)| \leq \frac{1}{\pi}\int_0^\pi |\exp((iX-Y)\sin v - isv)|\, dv = \frac{1}{\pi}\int_0^\pi \exp(-Y\sin v)\, dv \leq 1, \tag{3.6}$$

and it is not difficult to show that $\chi_s^{(1)}(Z) \sim$ const. $e^{iZ}/Z^{1/2}$ when $Z \to \infty$ in the upper-half Z-plane. Similarly in the lower-half Z-plane we use the conjugate function

$$\chi_s^{(2)}(Z) = \frac{1}{\pi}\int_0^\pi \exp(-iZ\sin v + isv)\, dv = J_s(Z) + i\mathbf{E}_s(Z). \tag{3.7}$$

It follows that

$$\int_0^\infty \frac{k+K}{k-K}\, k^n e^{-k(y+f)} J_s(k\rho)\, dk \tag{3.8}$$

$$= \frac{1}{2}\int_0^{\infty\exp(i\eta)} \frac{k+K}{k-K}\, k^n e^{-k(y+f)} \chi_s^{(1)}(k\rho)\, dk + \frac{1}{2}\int_0^{\infty\exp(-i\eta)} \frac{k+K}{k-K}\, k^n e^{-k(y+f)} \chi_s^{(2)}(k\rho)\, dk \tag{3.9}$$

$$+ \quad 2\pi i K^{n+1} e^{-K(y+f)} \chi_s^{(1)}(K\rho), \tag{3.10}$$

and the convergence of the resulting series for the potential can now be shown as for (2.7) above. Note, however, that products like

$$e^{-k(y+f)} \chi_s^{(1)}(k\rho) \cos s\alpha \tag{3.11}$$

are not solutions of Laplace's equation.

Another obvious approach is by means of Poisson's Integral which expresses the values of a potential inside a sphere as an integral over the values on the surface on the sphere. (An analogous argument was used in [Ursell 1950].) The bounds for derivatives which I have obtained by this method are adequate only when the radius of the sphere is sufficiently small and not for all values of $a < f$.

References

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