Non-uniqueness in the water-wave problem: an example violating the inside John condition

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1. Introduction

During the last decade uniqueness of the time-harmonic solution has been in the focus of much research in the linearized theory of water waves. A substantial breakthrough was the first example of non-uniqueness constructed by M. McIver (1996) in the two-dimensional water-wave problem. She applied the so-called inverse procedure which determines a physically admissible domain for a given solution instead of seeking a solution to the problem in a given domain. Developing this approach P. McIver & M. McIver (1997) obtained a non-uniqueness example for the axisymmetric water-wave problem, whilst Kuznetsov & Porter (1997) constructed a number of examples with different properties for the two-dimensional problem. Shortly after appearing the first non-uniqueness examples, one of the authors of the present work has proved the following uniqueness theorem for the two-dimensional problem (see Appendix in Linton & Kuznetsov 1997).

Let two surface-piercing bodies be immersed symmetrically about the y-axis in deep water and satisfy the inside John (IJ) *condition, that is, any vertical straight line through the portion of the free surface between the bodies, say $F_0 = \{-b < x < b,\ y = 0\}$, has no common points with the wetted bodies' contours.*

Then the homogeneous water-wave problem has only trivial symmetric (antisymmetric) solution, if the inequality

$$\pi\left(m + \frac{1}{4} \pm \frac{1}{4}\right) \le \nu b \le \pi\left(m + \frac{3}{4} \pm \frac{1}{4}\right) \tag{1}$$

holds with the sign + (−) for some $m = 0, 1, \ldots$.

This theorem means that the IJ condition is sufficient for uniqueness of symmetric/antisymmetric solution within the complementary intervals given by (1) for the non-dimensional spectral parameter νb. The examples constructed by Kuznetsov & Porter (1997), which include that of M. McIver (1996) as a particular case, show that this theorem can hardly be improved. The reason is that every interval where the symmetric solution is unique contains a subinterval of νb, for which there exists a two-body structure satisfying the IJ condition and trapping antisymmetric mode. The same result is shown to be true for the first three intervals where the antisymmetric solution is unique. Numerical calculations demonstrate that the same should be true for all intervals of νb, where (1) guarantees the uniqueness of antisymmetric solution.

The aim of the present work is to demonstrate that the IJ condition is not only sufficient, but also necessary for uniqueness in the intervals given by (1). We consider in detail the interval $(\pi/2, \pi)$, where the symmetric solution $u^{(+)}$ is unique, and outline how our approach works for other intervals. The idea of the proof is to construct a pair of bodies violating the IJ condition, trapping a symmetric mode and such, that $\nu b \in (\pi/2, \pi)$ for them.

2. Statement of the problem

The small-amplitude two-dimensional motion of an inviscid, incompressible fluid under gravity is considered. We assume the motion to be ω-periodic in time t and irrotational. Thus, it is described by a velocity potential $\mathrm{Re}\left\{u(x,y)\,\mathrm{e}^{-\mathrm{i}\omega t}\right\}$, where (x, y) are Cartesian coordinates with the origin in the mean free surface and the y-axis directed vertically upwards.

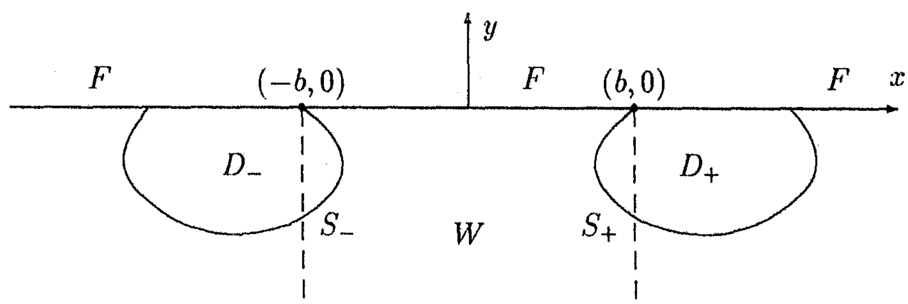

Figure 1: A definition sketch of the water domain.

Let $W = \{-\infty < x < +\infty,\ y < 0\} \setminus (D_+ \cup D_-)$ denote the domain occupied by water. We assume W to have infinite depth and to be symmetric about the y-axis (see fig. 1). Two rigid surface-piercing bodies D_+ and D_- are the mirror reflections of each other in the y-axis. The free surface is denoted by F and consists of three portions, two outside the bodies and one between them (it was referred to as F_0); the wetted boundary of $D_\pm$ is labelled $S_\pm$, and $S = S_+ \cup S_-$.

The eigenfunction u corresponding to a point eigenvalue ν (usually referred to as trapped mode solution) must satisfy the following homogeneous boundary value problem:

$$\nabla^2 u = 0 \quad \text{in} \quad W, \tag{2}$$

$$u_y - \nu u = 0 \quad \text{on} \quad F, \tag{3}$$

$$\partial u / \partial n = 0 \quad \text{on} \quad S, \tag{4}$$

and belong to the class of functions having the finite energy, that is,

$$\int_W |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}y + \nu \int_F |u|^2 \,\mathrm{d}x < \infty. \tag{5}$$

Without loss of generality, u satisfying (2)–(5) may be considered to be real.

3. Trapped mode solution violating the IJ condition

To formulate the main result we need two functions. We define the first of them as follows:

$$u(x,y) = (2\nu)^{-1}\left[G_x\left(x,y;-\pi/\nu,0\right) - G_x\left(x,y;\pi/\nu,0\right)\right], \tag{6}$$

where the two-dimensional Green function is given by the usual formula (see Wehausen &

Laitone 1960)

$$G(x,y;\xi,\eta) = -\log|z-\eta| + \log|z-\overline{\zeta}| + 2\int_{\ell_-} e^{k(y+\eta)}\,\frac{\cos k(x-\xi)}{k-\nu}\,dk,$$

$z = x + iy$, $\zeta = \xi + i\eta$, and ℓ_- denotes the contour going along the positive half-axis and indented below at ν. By the choice of the dipole points the integrals along indentations cancel in (6), and one immediately obtains that

$$u(x,y) = \frac{1}{\nu}\left[\frac{x+\pi/\nu}{(x+\pi/\nu)^2+y^2} - \frac{x-\pi/\nu}{(x-\pi/\nu)^2+y^2}\right] + \int_0^\infty \frac{\sin k(\nu x-\pi) - \sin k(\nu x+\pi)}{k-1}\,e^{k\nu y}\,dk,$$

where the integrand is bounded because the singularity in the denominator coincide with the zero of the numerator. Thus, u is a real harmonic function in the lower half-plane. Moreover, $u(x,y)$ is even with respect to x, and the free surface boundary condition holds for it on $\{x \neq \pm\pi/\nu,\ y = 0\}$. The last integral is bounded as $z \to \pm\pi/\nu$ as was shown by McIver (1996), and it decays as $|z| \to \infty$ as follows from Bochner (1959) *Lectures on Fourier Integrals*, §§ 2,5,8. Thus, u satisfies (5) in every fluid domain W, which does not contain a neighbourhood of the dipole points $(\pm\pi/\nu, 0)$.

The second required function is as follows:

$$v(x,y) = \frac{1}{\nu}\left[\frac{y}{(x+\pi/\nu)^2+y^2} - \frac{y}{(x-\pi/\nu)^2+y^2}\right] + \int_0^\infty \frac{\cos k(\nu x-\pi) - \cos k(\nu x+\pi)}{k-1}\,e^{k\nu y}\,dk,$$

that is, v is the streamfunction which corresponds to the velocity potential u, and has an arbitrary constant term to be equal to zero.

A family of fluid domains W, such that the IJ condition does not hold for W and u satisfies (2)–(5) in W can be constructed with the help of v. In fact, any streamline may be used

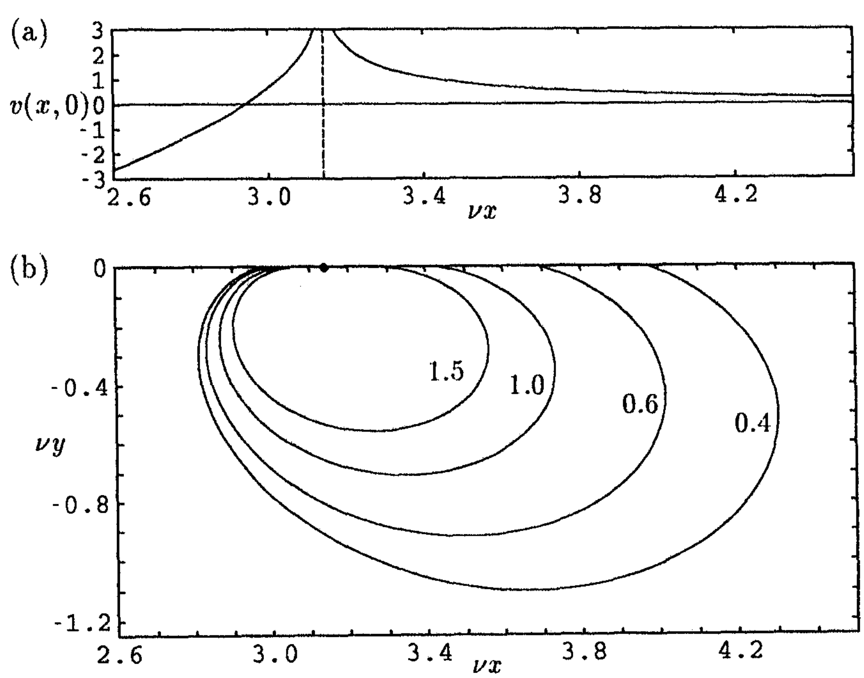

Figure 2: (a) the value of the streamfunction on $y = 0$, and (b) streamlines for v.

as S_+, if it has the following two properties. It connects with the positive x-axis on either side of the dipole point $(\pi/\nu, 0)$. The angle directed into W between the streamline and the positive x-axis is acute on the left of $(\pi/\nu, 0)$. On fig. 2(b) a number of streamlines defined by $v(x, y)$ and having these properties are plotted, and on fig. 2(a) the graph of $v(x, 0)$ is shown for convenience. Since $v(x, y)$ is an odd function with respect to x, the reflection of S_+ in the y-axis is also a streamline which we take as S_-. Now, let us formulate the main theorem concerning the existence of streamlines with these properties.

For every level $V > 0$ there exists only one streamline $S_+(V) = \{(x, y) : v(x, y) = V\}$ with all internal points in $\{x > 0,\ y < 0\}$ and the endpoints $(x_V^{(\pm)}, 0)$, such that $x_V^{(\pm)} > 0$, $\pm(x_V^{(\pm)} - \pi/\nu) > 0$, and $x_V^{(-)}\nu > 2\pi/3$. For every streamline $S_+(V)$ the IJ *condition does not hold.*

We note that $x_V^{(-)} = b$ for the water domain W having $S_+(V)$ and its reflection in the y-axis as the wetted rigid contours. Thus, we have $2\pi/3 < \nu b = \nu x_V^{(-)} < \pi$ for the defined W. Since u given by (6) delivers a symmetric eigenfunction satisfying (2)–(5) in this domain W, the immediate consequence of the main theorem is the following corollary:

The IJ *condition is necessary for the interval $(\pi/2, \pi)$ to be free of non-dimensional point eigenvalues νb corresponding to symmetric eigenfunctions.*

4. Concluding remarks

We restricted ourselves with the case of symmetric solution and of the uniqueness interval $\pi/2 < \nu b < \pi$, where $2b$ is the distance between two surface-piercing bodies along the free surface. Our choice is not a restriction, and has been made in order to be specific. For either symmetric and antisymmetric solution and for all intervals of uniqueness (1) examples of non-uniqueness, guaranteeing the necessity of the IJ condition, can be constructed. For this purpose the non-trivial potentials proposed by Kuznetsov & Porter (1997) should be modified in the same way as the potential of M. McIver (1996) has been modified in § 3.

Furthermore, the similar method works in the case of axisymmetric problem. Modifying the non-uniqueness example proposed by P. McIver & M. McIver (1997), one easily obtains that the IJ condition is necessary for the uniqueness theorem proved by Kuznetsov & McIver (1997) to be true in the axisymmetric problem.

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