Computing The Green Function for linear wave-body interaction

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The interaction between surface gravity waves and a structure in (or near) the free-surface is often analysed using potential theory, with linearised conditions applied on the body and the free-surface boundaries. Having assumed linearity, the response of the structure is described by a set of canonical radiation and diffraction response functions, which can be superposed with particular wave data to obtain particular solutions. These response functions are solutions to special distributions of normal velocity over the body which correspond to certain physical problems (*i.e.* forced motion of the body, or diffraction of a long-crested incident wave.) Another way of representing the interaction of waves with a structure is to compute The Green Function for the body. By "The Green Function" we refer to the particular Green function that satisfies the homogeneous form of the initial-boundary-value problem (including the body boundary condition) except at one singular point on the body surface (see [1]). Any desired quantity related to wave-body interaction may be expressed in terms of The Green Function. In general, the standard approach will be computationally more efficient than computing The Green Function, but there exist some situations where it may be advantageous to use a discrete form of The Green Function instead. Assume that a body is to be analysed which has J degrees of freedom, (6 rigid-body modes plus some number of flexible modes) and is subject to incident wave forcing from N_β heading angles. As explained below, computing the discrete form of The Green Function requires solving N hydrodynamic problems where N is the number of panels required to obtain converged results for the quantities of interest and a given body. Thus if the analysis is very complex, such that $J + N_\beta > N$, then computing The Green Function will be more efficient. Another situation where computing The Green Function might be attractive is when it is impractical (or impossible) to split the incident wave field into a finite sum of long-crested (uniform along one horizontal dimension) waves. For example, a body subject to incident waves which are diffracted and/or refracted by nearby corners or variable bottom topography.

The linear wave-body interaction problem can be expressed succinctly via the equations of motion in convolution form,

$$\sum_{k=1}^{J}(M_{jk} + a_{jk})\ddot{x}_k + b_{jk}\dot{x}_k + (C_{jk} + c_{jk})x_k + \int_{-\infty}^{t} d\tau\, K_{jk}(t-\tau)\dot{x}_k(\tau) = F_{jD}(t), \qquad j = 1, 2, ..., J. \tag{1}$$

In this expression, an over-dot indicates differentiation with respect to time. The body's inertia matrix is M_{jk}, and the hydrostatic restoring-force coefficients are given by C_{jk}. The force due to the radiation of waves by the body motion is expressed as a convolution of the radiation impulse-response functions K_{jk}, a_{jk}, b_{jk}, and c_{jk}; with the body velocity components in J degrees of freedom. The wave exciting force $F_{jD}(t)$ is typically taken to be a superposition of long-crested waves and is thereby expressed in the following convolution form

$$F_{jD}(t) = \int_0^{2\pi} d\beta\, F_{jD}(t,\beta) = \int_0^{2\pi} d\beta \int_{-\infty}^{\infty} d\tau\, K_{jD}(t-\tau,\beta)\zeta(\tau,\beta), \tag{2}$$

where $\zeta(t,\beta)$ is a time history of the elevation of the long-crested incident wave with heading angle β (the angle between the positive x-axis and the wave propagation direction) and $K_{jD}(t,\beta)$ is the impulse-response function for the diffraction force due to an impulsive long-crested wave from heading angle β. (In following seas with $U \neq 0$, there are three convolutions of this form which must be summed.) The diffraction force can also be expressed in terms of solutions to radiation problems via the Haskind-Newman relations

$$F_{jD}(t) = -\rho \int_{-\infty}^{t} d\tau \iint_{S_b} d\vec{\xi}\, \left[\dot{\phi}^I(\vec{\xi},\tau)\phi_{jn}^-(\vec{\xi},t-\tau) - \phi_j^-(\vec{\xi},t-\tau)\dot{\phi}_n^I(\vec{\xi},\tau)\right], \tag{3}$$

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where ϕ_j^- is the solution to the "reverse-flow" radiation problem (*i.e.* the radiation problem with the direction of the steady translation reversed.) Note that the Haskind-Newman relations provide the force, but no other information about the diffraction flow, and that a number of assumptions are involved in their derivation for $U \neq 0$, making them of limited appeal in that case.

If the incident wave is restricted to be time harmonic with frequency of encounter ω, amplitude $\mathcal{A}$, and heading angle β, then $\zeta(t) = \mathcal{A}\,\Re\{e^{i\omega t}\}$, and as $t \to \infty$ the response becomes $x_k(t) = \Re\{\xi_k(\omega,\beta)\,e^{i\omega t}\}$, and the equation of motion tend to

$$\sum_{k=1}^{J}\{-\omega^2[M_{jk}+A_{jk}(\omega)]+i\omega B_{jk}(\omega)+C_{jk}+c_{jk}\}\frac{\xi_k(\omega,\beta)}{\mathcal{A}} = X_{jD}(\omega,\beta); \qquad j=1,2,...,J. \tag{4}$$

The quantity $\xi_k/\mathcal{A}$ is usually called the response-amplitude operator (RAO). The frequency-response functions on the left-hand side of (4) (the added-mass and damping coefficients) are related to the radiation impulse-response functions through the Fourier transforms

$$A_{jk}(\omega) = a_{jk} - \frac{1}{\omega}\int_0^\infty dt\, K_{jk}(t)\sin\omega t; \quad B_{jk}(\omega) = b_{jk} + \int_0^\infty dt\, K_{jk}(t)\cos\omega t. \tag{5}$$

The frequency-response function on the right-hand side of (4) (the exciting force coefficient) is related to the diffraction impulse-response function through the Fourier transform

$$X_{jD}(\omega,\beta) = \int_{-\infty}^{\infty} dt\, K_{jD}(t,\beta)e^{-i\omega t}. \tag{6}$$

As in the time-domain, the diffraction force can be expressed in terms of radiation potentials via the Haskind-Newman relations

$$X_{jD}(\omega,\beta) = -i\omega\rho\iint_{S_b} d\vec{\xi}\,\left[\phi^I(\vec{\xi},\omega,\beta)\phi_{jn}^-(\vec{\xi},\omega) - \phi_j^-(\vec{\xi},\omega)\phi_n^I(\vec{\xi},\omega,\beta)\right]. \tag{7}$$

The physically motivated canonical radiation and diffraction problems defined above provide a complete picture of the linear interaction between waves and a structure. Another, perhaps less physically intuitive means of capturing this information is to compute The Green Function $\phi(\vec{x};\vec{\xi},t)$ for the body. This function satisfies the Laplace equation at every point in the fluid domain, the linear free-surface boundary condition on the free-surface boundary, and homogeneous Neumann conditions on the body boundary except at one singular point, thus

$$\vec{n}\cdot\nabla_{\vec{x}}\,\phi(\vec{x};\vec{\xi},t) = \delta(\vec{x}-\vec{\xi},t); \qquad \vec{\xi}\in S_b. \tag{8}$$

Any imaginable flow quantity can be expressed in terms of this function. For example, the corresponding first-order dynamic pressure impulse-response function (with $U=0$ for illustration purposes) is

$$p(\vec{x};\vec{\xi},t) = -\rho\,\dot{\phi}(\vec{x};\vec{\xi},t), \tag{9}$$

and the force impulse-response function is

$$F_j(\vec{\xi},t) = \iint_{S_b} d\vec{x}\; p(\vec{x};\vec{\xi},t)\,n_j(\vec{x}). \tag{10}$$

With these definitions, we can express the force on the body due to an arbitrary distribution of fluid velocity, $\vec{V}(\vec{x},t)$, as

$$F_j(t) = \iint_{S_b} d\vec{\xi}\int_{-\infty}^{\infty} d\tau\; \vec{n}(\vec{\xi})\cdot\vec{V}(\vec{\xi},\tau)\,F_j(\vec{\xi},t-\tau). \tag{11}$$

This expression is quite general and equally applicable to any distribution of normal velocity $\vec{n}\cdot\vec{V}$. For example, by setting $\vec{n}\cdot\vec{V} = n_1\delta(t)$ we can recover the six surge radiation impulse-response functions. Diffraction of an incident wave by the fixed body can be similarly represented. Consider an incident wave, ζ_I, with corresponding fluid velocity $\vec{V}_I(\vec{x},t)$ and (first-order) dynamic pressure $p_I(\vec{x},t)$. The diffraction force on the body can be split into two parts, $F_{jD}(t) = F_{jI}(t) + F_{jS}(t)$, where the first term

$$F_{jI}(t) = -\rho\iint_{S_b} d\vec{x}\; p^I(\vec{x},t)\,n_j \tag{12}$$

is often referred to as the Froude-Krilov force, and F_{jS} is the scattering force. Letting $\vec{V} = \vec{V}_I$ in Equation (11) gives the scattering force due to an arbitrary incident wave

$$F_{jS}(t) = \iint_{S_b} d\vec{\xi} \int_{-\infty}^{\infty} d\tau V_{In}(\vec{\xi},\tau)\, F_j(\vec{\xi}, t-\tau). \tag{13}$$

The same exercise may be carried out with a time-harmonic incident wave. In this case, let $\phi(\vec{x};\vec{\xi},t) = \Re\{\tilde{\phi}(\vec{x};\vec{\xi},\omega)e^{\imath\omega t}\}$ with

$$\vec{n}\cdot\nabla_{\vec{x}}\tilde{\phi}(\vec{x};\vec{\xi},\omega) = \delta(\vec{x}-\vec{\xi}), \qquad \vec{\xi}\in S_b. \tag{14}$$

Again, any flow quantity can be defined in terms of The Green Function. For example, the dynamic pressure frequency-response function is (again with $U = 0$ for illustration)

$$\tilde{p}(\vec{x};\vec{\xi},\omega) = -\imath\omega\rho\,\tilde{\phi}(\vec{x};\vec{\xi},\omega), \tag{15}$$

and the force frequency-response function is

$$\tilde{F}_j(\vec{\xi},\omega) = \iint_{S_b} d\vec{x}\, \tilde{p}(\vec{x};\vec{\xi},\omega)\, n_j(\vec{x}), \tag{16}$$

which gives the corresponding general expression for the force on the body due to the fluid velocity $\vec{V}(\vec{x},\omega)$

$$\tilde{F}_j(\omega) = \iint_{S_b} d\vec{\xi}\, \vec{n}(\vec{\xi})\cdot\vec{V}(\vec{\xi},\omega)\, \tilde{F}_j(\vec{\xi},\omega). \tag{17}$$

Similarly, the added-mass, damping, and long-crested wave exciting force coefficients can be recovered by considering the appropriate distributions of $\vec{n}\cdot\vec{V}$.

To demonstrate the practical application of The Green Function, we compute it for a bottom mounted circular cylinder, and then use it to recover the diffraction force due to long-crested incident waves. The accuracy is then compared to a direct solution of the canonical diffraction problem. The calculations are made using the low-order panel method program WAMIT. In the context of a low-order (constant strength) panel method, the discrete analogue to the boundary conditions on ϕ and $\tilde{\phi}$ are

$$\vec{n}\cdot\nabla\phi_{jk}(t) = \left\{ \begin{array}{ll} \delta(t); & j=k \\ 0; & j\neq k \end{array} \right\} \quad j=1,2,...,N;\ k=1,2,...,N; \tag{18}$$

and

$$\vec{n}\cdot\nabla\tilde{\phi}_{jk}(t) = \left\{ \begin{array}{l} 1;\ j=k \\ 0;\ j\neq k \end{array} \right\} \quad j=1,2,...,N;\ k=1,2,...,N; \tag{19}$$

which can be thought of as N special generalised radiation problems.

Figure 1 shows the magnitude of the horizontal wave exciting force on the cylinder as a function of frequency, while Figure 2 shows the absolute error in the two calculations. Both calculations where made using $N = 252$ panels at 120 evenly spaced frequencies. Using The Green Function produces results of comparable, although typically slightly lower accuracy. This is not surprising since using The Green Function requires another set of integrations over the body surface which can be expected to introduce additional errors into the calculations.

Acknowledgments

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References

[1] P.M. Morse and H. Feshbach. *Methods of Theoretical Physics.* McGraw-Hill, New York, 1953.

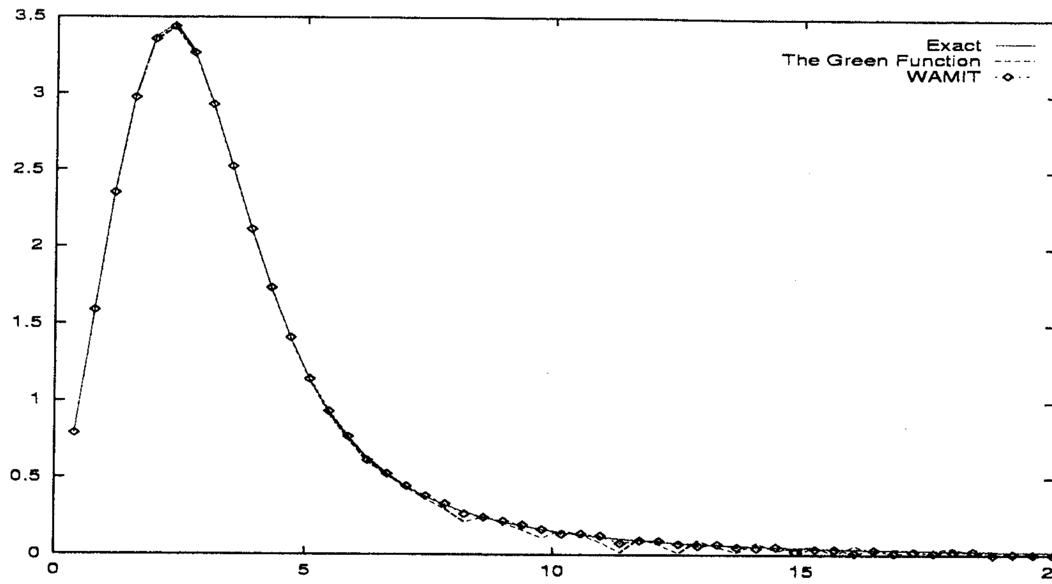

Figure 1: Magnitude of the non-dimensional surge exciting force $\frac{X_1}{\rho g R^2 A}$ for a bottom mounted circular cylinder of radius R, in water of depth $H = 1$, plotted against $(\frac{g}{R})^{\frac{1}{2}}\omega$.

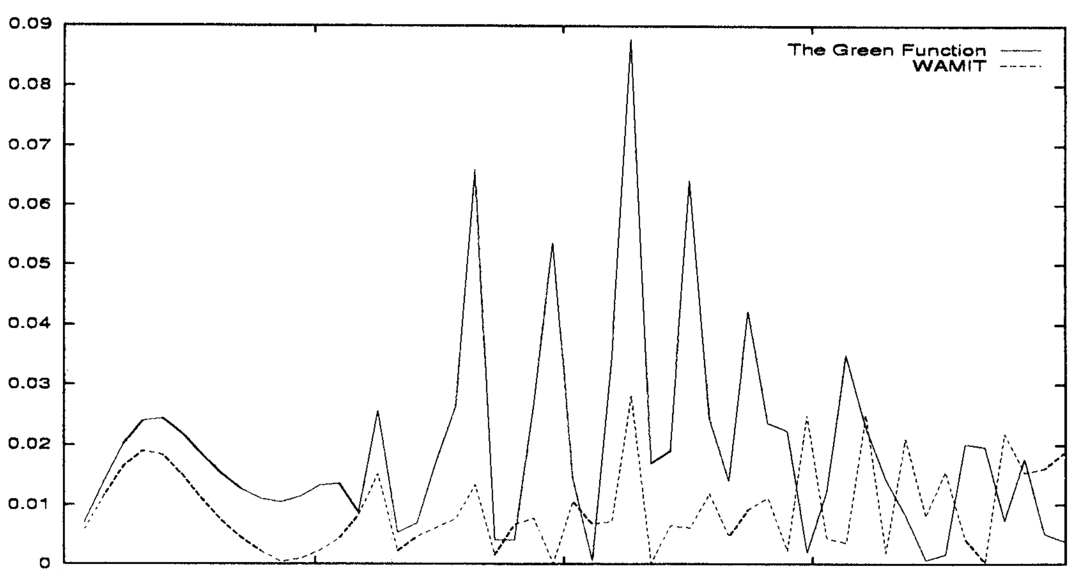

Figure 2: Absolute error of the two methods in the magnitude of the surge exciting force.