Wavelet and spline methods for the solution of wave-body problems

by

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1 Introduction

Integral equation methods represent powerful alternatives in computation of potential flow around geometries and bodies, where an important example is interaction between water waves and floating bodies. Numerical implementation of the integral equation has often been based on a low-order method, where the boundary of the geometry is subdivided into piecewise straight lines in two dimensions or quadrilaterals in three dimensions. The unknown potential or source-strength is assumed to be constant over each subdivision of the boundary. For complex geometries like e.g. the wetted part of an oil platform, this method leads to a large number of unknowns (n), if a reasonable accuracy of the potential and the flow shall be obtained.

The rather extensive applications of the low-order method illustrate its power. It is, however, desirable to investigate higher order integral methods which have features not included in a low-order method: possibility of finding derivatives of the potential, reduction of the number of unknowns and thereby the size of the matrices, fast convergence of the method, and adaptivity. Another aspect relates to geometrical design. Most practical geometries today are designed by advanced mathematical procedures, e.g. using splines. It is therefore desirable to make available wave analysis tools which are based on the same mathematical procedures as in the modelling of the geometry. The purpose is to integrate efficient and accurate computations of the flow and forces in the design process.

We investigate wavelet and spline methods, which have rather different properties, see Nygaard *et al.* (1996). One of the advantages of the wavelet method is the possibility of performing compression of the coefficient matrix of the system. According to Beylkin, Coifman and Rokhlin (1991), it is possible to devise an $\mathcal{O}(n)$ algorithm for certain integral operators, where n is the number of unknowns. We test the methods on Fredholm integral equations of the second kind. Preliminary results for the wavelet method show that the order of convergence for the present integral operator depends on the geometry. We compare the wavelet and spline methods. The latter method has, in the context of wave analysis, been discussed by Lee *et al.* (1996).

For simplicity we assume two-dimensional motion and consider a half-immersed rectangular cylinder floating in a free surface, responding to incoming waves. Coordinates (x, y) are introduced, with x being horizontal and y vertical. Assuming time harmonic motion with frequency ω, the potential is on the form $\Phi = \mathrm{Re}(\hat{\chi} e^{i\omega t})$, where $\hat{\chi}$ satisfies the Laplace equation in the fluid domain, $\partial\hat{\chi}/\partial y = K\hat{\chi}$ at $y = 0$ ($K = \omega^2/g$), radiation conditions in the far field and $\partial\hat{\chi}/\partial n = V_n$ at the contour S of the cylinder, n is the inward pointing normal vector. From Green's theorem we obtain the usual integral formulation by

$$\int_S \left(\hat{\chi}(\boldsymbol{\xi}) \frac{\partial G(\boldsymbol{\xi}, \boldsymbol{x})}{\partial n_\xi} - \frac{\partial \hat{\chi}(\boldsymbol{\xi})}{\partial n_\xi} G(\boldsymbol{\xi}, \boldsymbol{x}) \right) dS_\xi = \begin{cases} -\pi \hat{\chi}(\boldsymbol{x}) & \text{on } S \\ -2\pi \hat{\chi}(\boldsymbol{x}) & \text{in the fluid} \end{cases} \tag{1}$$

where $r, r' = [(x-\xi)^2 + (y \mp \eta)^2]^{1/2}$,

$$G(\boldsymbol{\xi}, \boldsymbol{x}) = \ln r - \ln r' + 2\mathrm{Re}\left(\mathrm{e}^Z \int_\infty^Z \frac{\mathrm{e}^{-w}}{w} dw \right) + 2\pi i \mathrm{e}^Z$$

and $Z = -iK(\xi - x) + K(\eta + y)$, $-3\pi/2 < \arg Z < -\pi/2$.

2 The wavelet method - multiresolution analysis

The wavelet method is a Galerkin scheme with a basis which decomposes functions into pieces of different frequency content locally in space. We expand a function f as

$$f = f_0 + g_0 + g_1 + \cdots = \sum_k c_k^0 \phi_k^0 + \sum_k d_k^0 \psi_k^0 + \sum_k d_k^1 \psi_k^1 + \cdots \tag{2}$$

where the basis consists of the functions ϕ_k^0 and ψ_k^j. The subscript indices denote a translation in space (k), and the superscript indices give the location of the frequency (j). The translation in space is uniform, so the function is defined on a uniform grid. Where the function is reasonably smooth, the frequency content will be concentrated. This means that the coefficients corresponding to this particular localization in space will be dominant for a few frequencies. Correspondingly, frequencies which are not so dominating will have small coefficients. Depending on the regularity of the function and necessary accuracy of approximation, a number of these coefficients may be discarded. This makes it feasible to design an adaptive procedure for the solution of our problem, based on a hierarchical structure of *multiresolution analysis.* In short, the term *multiresolution analysis* is coined for the collection of nested approximation spaces spanned by the functions ϕ_k^0 and ψ_k^j. For details on the multiresolution analysis, see e.g. Jawerth and Sweldens (1994). We use the Daubechies wavelet basis indexed by the number N, as in Daubechies (1992). We use $N = 3$ and $N = 8$. The scaling and wavelet functions for $N = 3$ is depicted in fig. 1.

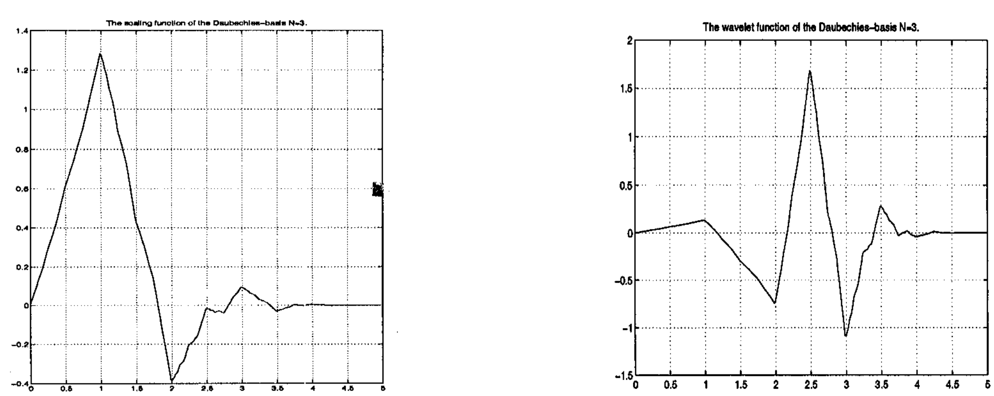

Figure 1: Scaling and wavelet function

3 Spline method

Using splines, we get the simplicity of piecewise polynomials, and for many applications, the geometry will also naturally be defined by splines. This is the case for our examples. The potential is represented by the linear combination of a number of B-splines B_i, for $i = 1, \ldots, m$. In our case, the knot-vector will always be a refinement of the knot vector for the geometry, but that is no requirement. A number of projection schemes are appropriate to use, and we start by inserting the expansion $\chi(t) = \sum_{i=1}^{m} \chi_i B_i(t)$ into the parameterized integral equation. **Collocation** with five collocation points between each knot gives an overdetermined system which is solved by a least-squares method. For the **Galerkin** case we use the B-splines both as trial and test functions, and multiply (1) (after parameterization and application of the spline-expansion,) by $B_i(t)$ for all i, and integrate along the contour. This leads to a square system of exactly m equations.

4 Numerical results

We have implemented the methods and compared them with respect to accuracy (L^2-error) and the corresponding number of non-zero matrix elements of the linear systems. The problems are well conditioned, and the systems can be solved by an iterative method utilising only matrix-vector multiplications, with a constant number of iterations.

Our results show that

- The wavelet method results in a matrix which may easily be compressed, resulting in a very sparse system yielding an accurate solution.
- For smaller problems, or when high accuracy is not needed, a spline implementation will be both simple and efficient.
- For larger problems, e.g. when the geometry is complicated or a high degree of accuaracy is needed, an efficient implementation of the wavelet method will be able to outperform the spline method.

We show an example for the case of a square cylinder in the long wave approximation ($K = 0$) in fig. 2. The figure illustrates that large parts of the coefficient matrix in the wavelet method may be discarded, and that a higher accuracy is obtained in the wavelet case than in the spline case. In this particular example we also compare with analytical results by using the Schwarz-Christoffel transform. We find convergence, also at the corners of the square cylinder. Further results, for different wave frequencies, will be presented at the workshop.

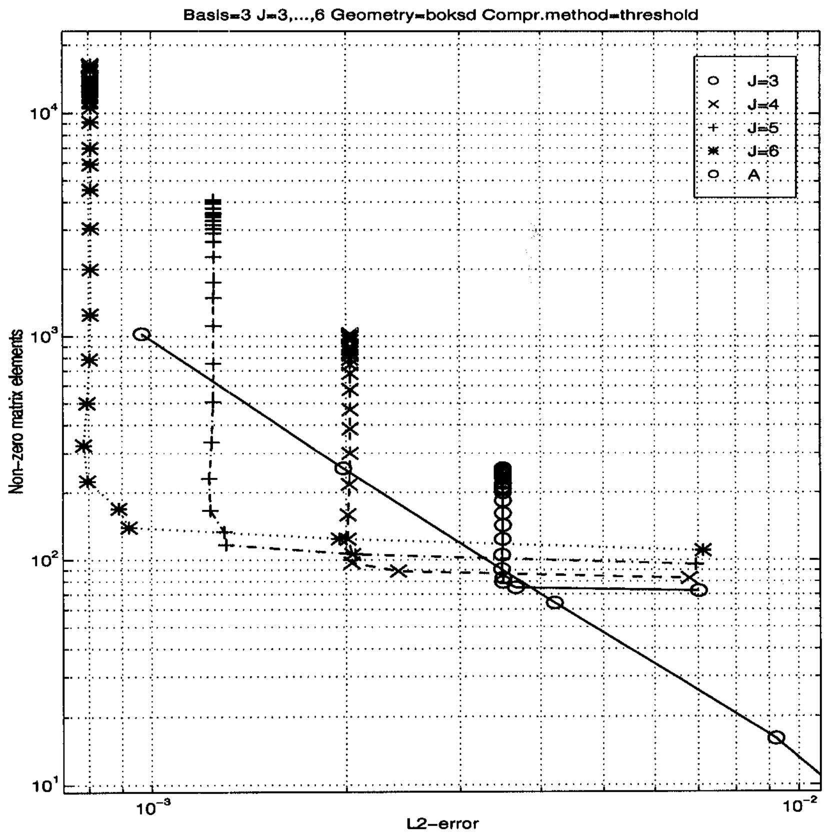

Figure 2: Accuracy of potentials for half-immersed square cylinder, surge motion, no incoming waves. $K=0$. Solid line: spline-Galerkin solution with quadratic splines. Dashed and dotted lines: wavelet solutions, basis $N=3$, varying degrees of matrix-compression. The number of unknowns before compression is 2^{J+1}, $J=3,4,5,6$. Horizontal axis: L^2 error (accuracy). Vertical axis: Number of non-zero matrix elements in the linear system.

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DISCUSSION

Huang J.:

1) Normally, when a function $f(x)$ is expanded in wavelet space, it is expanded in one wavelet space $\{V_m\}$, i.e. at scale m. Why did you expand $f(x)$ (as shown in eq. 2) using different $\{V_m\}$ and superimpose them?

2) You showed the results of potential derivative, did you involve the direct evaluation of wavelet in your computation? The derivative of Daubechies wavelet is highly oscillated.

Nygaard J.O., Grue J.:

1) The functions decomposed into multiresolution analyses are indeed decomposed only in one space V_J (a space spanned by translates of the *scaling function* ϕ,) but it is then decomposed further into the *wavelet spaces* (spanned by translates and dilations of the actual *wavelet* ψ) W_j, for $j = J_o, \ldots, J-2, J-1$, together with a remainder in V_o. Here, $V_J = U_{j=0}^{J-1} W_j \cup V_0$. (Note that V_0 is just a convenient way of denoting the coarsest space where the sequence of nested spaces is truncated.)

2) No, the direct evaluation of the scaling function ϕ or the wavelet function ψ were not used at any stage. The Daubechies wavelets (and scaling functions) are indeed highly oscillating for large N, and they are not very smooth for small N. This carries over to the derivatives of the functions, but we note that there are Daubechies bases with arbitrarily smooth scaling functions avd wavelets, and therefore also arbitrarily smooth derivatives. However, there is a connection between the smoothness and the oscillatory behaviour. (As well as length of support, length of discrete filters and so on, so any choice of N will be a compromise.)

When, for final plots and other uses of the functions expanded in the wavelet bases, evaluations are needed, the recursive refinement scheme (also denoted the *pyramid scheme*,) gives a stable and efficient way of obtaining large numbers of evaluations of the functions. This applies also to the more irregular of the Daubechies bases.

Magee A.:

1) In the compression method, you must search through the matrix to find the smallest value. Is this a significant computational burden?

2) Your results for the spline method seem to indicate that the results are improved with higher discretization (that is, the irregular behaviour is reduced for finer discretization). But is the exact (theoretical) irregular frequency equal to one of those used in the calculations? Have you checked frequencies nearby to be sure you are not missing the most irregular behavior of the numerical solutions which may change as a function of the discretization?

Nygaard J.O., Grue J.:

1) Yes, this is a burden in our implementation. However, this is done in this particular way because we have wanted to investigate whether or not the wavelet method will be able to compete with methods based on splines before putting effort into developing more efficient code. For an efficient implementation, larger portions of the elements to be discarded in the compression process have to be predicted without their actual computation.

2) Irregular frequencies are always present in the formulation, however, for a successively finer discretization of the spline knot vector, we find that the ill-behaved frequency domain is reduced. We shall look into more detail regarding the dependence of the observed irregular frequencies on the discretization.