# Recent progress in dealing with the singular behavior of the Neumann-Kelvin Green function

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The Neumann-Kelvin formulation of the linear wave-resistance problem is considered. Due to the singular behavior of the Neumann-Kelvin Green function, special care is required when dealing with surface piercing bodies. Consequently, an element integration technique is proposed as a discretization paradigm. This method, which alleviates the singular behavior of the Green function, is implemented within the frame of a bounded domain formulation for the Neumann-Kelvin problem. Advantages and drawbacks are presented, and possible improvements discussed.

## A bounded domain formulation for the Neumann-Kelvin problem

We consider the wave-resistance problem of a body moving at constant speed, $-U_0\,\vec{x}$, in the half space $z \leq 0$ occupied by an ideal fluid at rest. In the Neumann-Kelvin approach, the velocity potential is decomposed, in the co-moving reference frame, as the sum $U_0\,x + \varphi_e$, where $\varphi_e$ is solution of the Neumann-Kelvin problem.

Panel methods making use of the Neumann-Kelvin Green function, define the perturbation potential $\varphi_e$ in terms of dipole and source distributions over the body boundary $\Gamma$. The potential $\varphi_e$, which has then the following integral representation:

$$\varphi_e(M) = \int_\Gamma [\varphi_e(P)\,\partial_{\vec{n}_P} G_\nu(M,P) - f(P)\,G_\nu(M,P)]\,d\Gamma_P, \tag{1}$$

where $G_\nu$ is the Neumann-Kelvin Green function, $\nu = g/U_0^2$ and $f(P) = -U_0\,(\vec{n}_P \cdot \vec{x})$, is obtained as solution of an integral equation on $\Gamma$.

However, rather than solving this integral equation, we here consider a bounded domain problem which is derived using a variational formulation/integral representation coupling method [4]. In addition to its theoretical interest, this approach presents some practical advantages: for example, equation (1) can now be defined on an arbitrary coupling surface $\Sigma$ thus avoiding the $1/r$ singularities of the Green function. Furthermore, in order to avoid computing second order derivatives of $G_\nu$, the bounded domain formulation is herein modified by introducing a potential $\tilde{\varphi}_i$, solution of a Dirichlet problem in $\Omega_i$, an interior domain of the body (Figure 1-a). Consequently, solving the Neumann-Kelvin problem for a submerged body is shown to be equivalent to finding the solution $(\varphi_e, \tilde{\varphi}_i)$ of the problem:

$$\text{(2)}\begin{cases} \displaystyle\int_{\Omega_e} \nabla\varphi_e \cdot \nabla\bar{\psi}_e - \frac{1}{\nu}\int_{\widehat{SL}} \partial_x\varphi_e\,\partial_x\bar{\psi}_e\,dS + \mu\int_\Sigma \varphi_e\,\bar{\psi}_e\,d\Sigma & \displaystyle\int_\Gamma f\bar{\psi}_e\,d\Gamma \\ \displaystyle+\int_{\Omega_i} \nabla\left(r_\Gamma^{F_a}(\varphi_e) + \tilde{\varphi}_i\right)\cdot\nabla\overline{\tilde{\psi}_i} & = \displaystyle -\int_\Sigma \bar{\psi}_e \int_\Gamma f(P)\,G_\mu\,d\Gamma_P\,d\Sigma_M \\ \displaystyle-\frac{1}{\nu}\int_\sigma \bar{\psi}_e \int_{\Omega_i} \nabla\left(r_\Gamma^{F_a}(\varphi_e) + \tilde{\varphi}_i\right)\cdot\nabla\left(r_\Gamma^{F_a} + r_{F_a}^\Gamma\right)(G_x)\,d\sigma & \displaystyle+\frac{1}{\nu}\int_\sigma \bar{\psi}_e \int_\Gamma f(P)\,G_x\,d\Gamma_P\,d\sigma_M \\ \displaystyle+\int_\Sigma \bar{\psi}_e \int_{\Omega_i} \nabla\left(r_\Gamma^{F_a}(\varphi_e) + \tilde{\varphi}_i\right)\cdot\nabla\left(r_\Gamma^{F_a} + r_{F_a}^\Gamma\right)(G_\mu)\,d\Sigma & \end{cases}$$

for any test function $(\psi_e, \tilde{\psi}_i)$. In this formulation, the conventions are:

- $G_\mu(M,P) = (\partial_{\vec{n}_M} \cdot + \mu\cdot) G_\nu(M,P)$, where $\mu$ is a complex number of non-zero imaginary part so as to avoid irregular frequencies;
- $G_x(M,P) = \partial_{x_M} G_\nu(M,P)\,(\vec{n}' \cdot \vec{x})$, where $\vec{n}'$ is the vector lying in $\widehat{SL}$ and normal to $\sigma$ at $M$;
- $r_\Gamma^{F_a}(\psi)$ is such that $r_\Gamma^{F_a}(\psi) = \psi$ on $\Gamma$ and $r_\Gamma^{F_a}(\psi) = 0$ on $F_a$, and conversely for $r_{F_a}^\Gamma$.

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Unfortunately, because of the singular behavior of $G_\nu(M,P)$ for $P$ downstream of $M$, the latter being on the free surface (see [7]), the equivalence between the Neumann-Kelvin problem and (2) could not be established for surface piercing bodies. In view of this difficulty, we shall restrict ourselves to the devising of a discretized formulation of (2) for submerged bodies which remains numerically well behaved in the limit of zero depth.

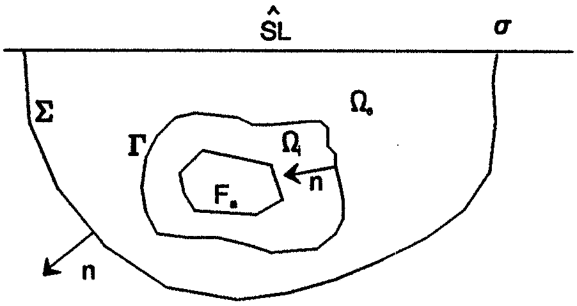


FIG. 1-a: *Coupling method*

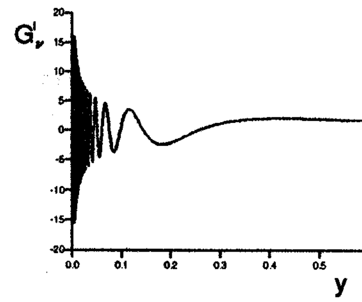


FIG. 1-b: *Singular behavior of $G^l_\nu$*

For this purpose, the bounded domain formulation is solved numerically using a finite element method: the different terms in (2) are discretized with the help of basis functions $w_l$ which depend on the volume discretization of the domains $\Omega_i$ and $\Omega_e$, and on the type of interpolation functions being used. The singular behavior of the Green function forbids however the use of classical discretization techniques for the terms of (2) which involves $G_\nu(M,P)$ at its singular regime [1]. Consequently, a specific discretization method must be devised.

## An element integration method

The Green function $G_\nu$ can be decomposed as the sum of a near-field and a far-field component, $G^l_\nu$, the latter accounting entirely for its singular behavior. The difficulties which arise when discretizing the terms involving $G^l_\nu$ in (2), can be circumvented by first, interchanging the orders of integration between the points $M$ and $P$ in (2), then, performing analytically the spatial integration with respect to $M$. An approach following this principle has also been proposed for the diffraction-radiation problem with forward speed: see [6]. The present procedure leads to computing analytically the integrals:

$$(3) \qquad \int_\sigma \bar{w}_l\, \partial_{x_M} G^l_\nu(x,y,z')\,(\vec{n}'_M\cdot\vec{x})\; d\sigma_M \quad \text{and} \quad \int_\Sigma \bar{w}_l\, (\partial_{\vec{n}_M}\cdot + \mu\cdot)\, G^l_\nu(x,y,z')\, d\Sigma_M$$

with the notations $x = x_P - x_M$, $y = y_P - y_M$, and $z' = z_P + z_M$. In performing this task, we benefit here from the ability of choosing an arbitrary coupling surface $\Sigma$. Hence by imposing $\Omega_e$ to be a rectangular prism (Figure 2-a), analytical integration of (3) is only required for $M$ on $\sigma_1$ and $\Sigma_1$—the portions of $\sigma$ and $\Sigma$ directly upstream of the body.

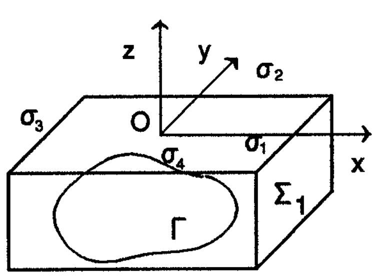


FIG. 2-a: *Particular choice of $\Omega_e$*

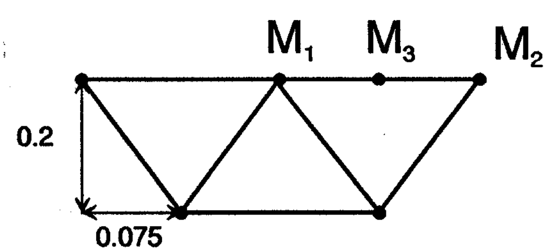


FIG. 2-b: *Discretization of $\Sigma_1$*

Based on experience with computations for submerged bodies, Lagrange elements of degree 2 are retained as they provide satisfactory rates of convergence for a limited number of unknowns. Therefore, evaluating the integrals in (3) for a triangular element $T = (M_1, M_2, M_3)$ of $\Sigma_1$ is equivalent to computing the element integrals:

$$
(4)\qquad \begin{cases} T_j = \displaystyle\int_{[M_1,M_2]} w_j\, \partial_{x_M} G^l_\nu(x,y,z')\, d\sigma_M & j = 1,2,3 \\ T^r_j = \displaystyle\int_T w_j\, \partial_{x_M} G^l_\nu(x,y,z')\, d\Sigma_M\,, \quad T^i_j = \displaystyle\int_T w_j\, G^l_\nu(x,y,z')\, d\Sigma_M & j = 1..6\,. \end{cases}
$$

For this purpose, we retain the representation of $G^l_\nu$ used in [2]. We are thus led to consider the following complex contour integrals:

$$
(5)\qquad G_k(r,\alpha,\xi) = \int_{L_+} \frac{\exp\left[-\frac{r}{2}\cosh(2u - i\alpha) + i\xi\cosh u\right]}{(\cosh u)^{k-1}(\sinh u)^{k+1}}\, du\,, \text{ for } k = 0..2,
$$

$$
(6)\qquad E^q_{k,k'}(r,\alpha,\xi,Q) = \int_{L_+} \frac{\exp\left[-\frac{r}{2}\cosh(2u - i\alpha) + qu + i\xi\cosh u\right]}{(\cosh u)^k\,(z_Q\cosh u - iy_Q\sinh u)^{k'}}\, du\,, \text{ for } \begin{cases} q = -1,0,1 \\ k = 0,1,2 \\ k' = 1,2,3 \end{cases},
$$

with $\xi = \nu|x|$, $r = \nu\sqrt{y^2 + z'^2}$, $\alpha = \arctan(-y/z')$, and where $L_+$ is a path joining $-\infty$ to $+\infty$ and avoiding the poles of the integrand. As the integrals $G_k$ and $E^q_{k,k'}$ are similar to the expression of $G^l_\nu$, the various approximations described in [2] are extended to the present case. Two complementary approximations per integral are thus derived which provide numerical results with an absolute accuracy of at least five significant digits, and this for $\xi$ in a range sufficiently large for the present applications. These approximations consist in: *a)* convergent series expansions for values of the parameter $\mathcal{M} = \xi^2/4r \leq 16$, and *b)* asymptotic expansions along with highly oscillatory integrals when $\mathcal{M} \geq 16$. These oscillatory integrals, similar to that introduced in [8], are evaluated following [5]. The main difference between $G^l_\nu$ and the functions $G_k$ and $E^q_{k,k'}$ lies in the fact that the latters are defined and continuous for $\xi > 0$, $r = 0$, $|\alpha| = \pi/2$, whereas the former is singular there.

# Applications

## Submerged ellipsoid

The present element integration approach has been compared, for the case of a submerged ellispoid, with a classical discretization method as well as with the semi-analytical results of Farell [3]. Wave-resistance results show good agreement between the element integration method and Farell's results for an ellipsoid with an aspect ratio of 5 at a submergence depth of a quarter of the focal distance: see Figures 3-a, b.

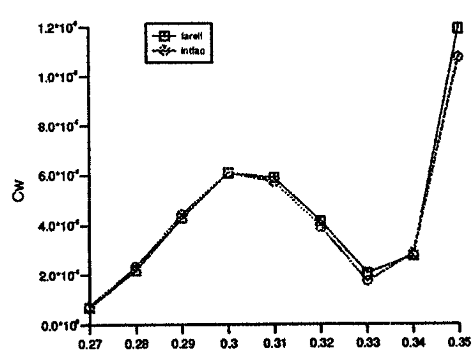


FIG. 3-a: *Small Froude number*

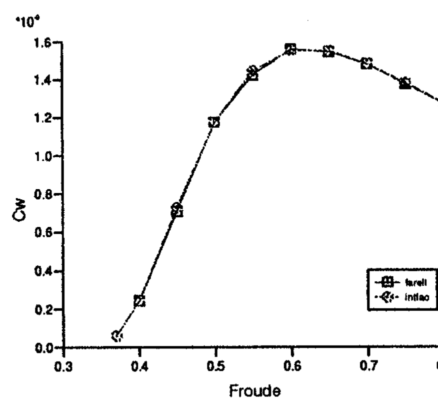


FIG. 3-b: *Large Froude number*

## Surface piercing ellipsoid

Computations with the same ellipsoid, but now half submerged, have been performed. Wave-resistance results appear to be strongly unstable with respect to mesh refinements. An analysis of the line and surface integral contributions in (3) furnishes a possible explanation for this behavior. Indeed these contributions present oscillations near the tracks of the discretization points lying on $\sigma_1$. These oscillations, which cannot be resolved with a reasonably fine mesh, render the element integrals $T_j$ acutely sensitive to the location of the point $P$. This peculiarity is illustrated in Figures 4-a, b for the element integrals $T_1$ and $T_1^r$ associated to the point $M_1$ of coordinates $x = 1.4$, $y = z = 0$ (see Figure 2-b): significant peaks are clearly visible about the points $y_P = 0, \pm 0.075$. The fact that these peaks are more pronounced for $T_1$ than for $T_1^r$, and that their magnitude increases with the distance $\xi$, indicates that they are inherent to the highly oscillatory behavior of $G_\nu^l$.

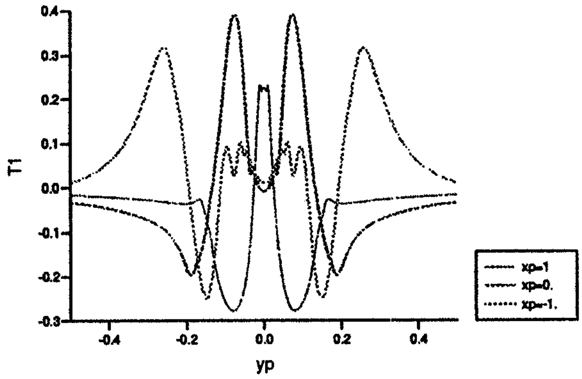


FIG. 4-a: *Behavior of* $T_1$

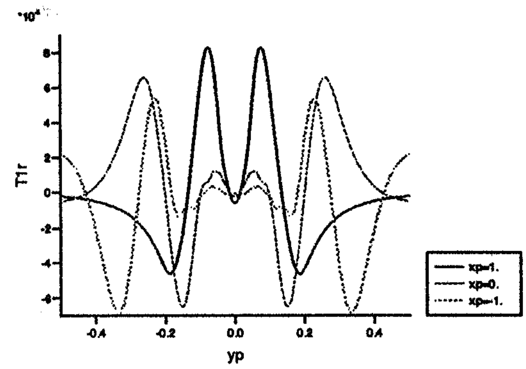


FIG. 4-a: *Behavior of* $T_1^r$

# Discussion

Analytical evaluations of the line and surface integral contributions in (3) has alleviated the singular behavior of $G_\nu^l$, thus resulting in a proper numerical discretization of the bounded domain formulation (2). However, at this stage, numerical results could not be obtained for surface piercing bodies due to the strongly oscillatory behaviors of the element integrals $T_j$. Such behaviors are associated with the discretization of the boundary $\Sigma_1$ and the particular choice of basis functions with discontinuous slopes. Significant improvements could be achieved in several ways, namely:

- by performing analytically the spatial integration with respect to the field point $P$: while this task does not present further difficulties, the required analytical computations are significantly heavier.
- through the use of $C^m$ elements, $m > 0$: the current finite element procedure based on Lagrange element would need to be taylored to such a case.
- through the use of spectral elements: substantial work would be needed to devise a practical method capable of handling an arbitrary shaped boundary such as $\Gamma$.

# References


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# DISCUSSION

**Kuznetsov. N.:** The finite element method usually leads to tridiagonal matrix while matrices arising from discretization of boundary integral equations are complete. Does the matrix in your coupled approach have the advantage of the FEM to be tridiagonal?

**Doutreleau Y., Clarisse J-M.:** No, it's not the case because of the coupling terms between the hull $\Gamma$ and the coupling surface $\Sigma$. So we have more unknowns than in boundary integral method, but not so many because in many problems, only one layer of finite elements is needed. The real advantage of the coupling method consists in involving no singularities of Rankine type in the Green function.
The second advantage in the precise problem involved in this talk is that we can decrease the analytical work drastically by choosing an appropriate coupling surface.