Solitary waves with algebraic decay

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1 Introduction

The study of capillary–gravity solitary waves is fairly recent. Among these waves, the most relevant from a physical point of view are the waves which bifurcate from a train of infinitesimal periodic waves with the property that their phase and group velocities are equal. These waves exist both in finite and in infinite depth. However, their properties differ with the depth. These waves were first computed numerically in infinite depth by Longuet–Higgins (1989). A physical interpretation was provided simultaneously by Akylas (1993) and Longuet–Higgins (1993). They showed that these waves correspond to stationary solutions of the nonlinear Schrödinger equation that governs slow modulations in space and in time of capillary–gravity waves. In certain regions of parameter space, it is well-known that the nonlinear Schrödinger equation (nlS) admits solutions in the form of wave packets, characterized by two length scales, the length of the envelope and the wavelength of the oscillations inside the envelope. The envelope travels at the group velocity while the oscillations travel at the phase velocity. It is therefore natural to obtain steady wave packets when phase and group velocities are equal. Additional numerical results were provided in infinite depth by Vanden–Broeck & Dias (1992) and in finite depth by Dias, Menasce & Vanden–Broeck (1996).

The important difference between finite and infinite depth comes from the properties of the dispersion relation. Let h denote the depth, k the wave number, g the acceleration due to gravity, c the phase velocity, σ the coefficient of surface tension. In finite depth, the dispersion relation is

$$c^2 = \left(\frac{g}{k} + \sigma k\right) \tanh(kh)\,, \tag{1.1}$$

while in infinite depth it becomes

$$c^2 = \left(\frac{g}{|k|} + \sigma|k|\right)\,, \tag{1.2}$$

which is singular at $k = 0$. This singularity leads to nonlocal terms, which are not present in the nlS equation. In the context of modulated waves, these nonlocal terms represent the interaction between the wave envelope and the induced mean flow. For gravity waves, they were computed first by Dysthe (1979) and recomputed by Stiassnie (1984) by using the so-called Zakharov's equations. Hogan (1985) extended Stiassnie's analysis to capillary–gravity waves. In this abstract, we construct an analytical solution of Hogan's equation which shows that the presence of the nonlocal terms leads to an algebraic decay in $1/x^2$ of the solitary waves. Note that Longuet–Higgins (1989) predicted such a decay purely on physical grounds and that the numerical results of Vanden–Broeck & Dias (1992) also show such a decay.

2 Analytical results

Let the free surface elevation be described by

$$\eta(x,t) = \tfrac{1}{2}\left\{A(x,t)\mathrm{e}^{i(kx-\omega t)} + \text{c.c.}\right\}. \tag{2.1}$$

Using as a basis the so-called Zakharov's equations, Hogan (1985) derived a higher-order nlS equation, similar to Dysthe's equation which was obtained by the method of multiple scales. Dimensionless variables (denoted with primes) are introduced by taking $\sigma/\rho c^2$ as unit length $[L]$ and $\sigma/\rho c^3$ as unit time $[T]$. In addition we introduce a small parameter ϵ (see below) as well as slow variables:

$$(x,z) = [L](x',z'), \quad t = [T]t', \quad A = \epsilon[L]A', \quad \overline{\phi} = \epsilon^2\frac{[L]^2}{[T]}\overline{\phi}', \quad (X,Z,T) = \epsilon(x',z',t').$$

ϕ is the velocity potential, z the vertical coordinate. From now on, the primes will be dropped. The amplitude A satisfies the equation

$$A_T + c_g A_X - \mathrm{i}\epsilon(pA_{XX} + q|A|^2A) + \epsilon^2(rA_{XXX} + uA^2A_X^* + v|A|^2A_X) + \mathrm{i}\epsilon^2 kA\overline{\phi}_X|_{Z=0} = 0, \tag{2.2}$$

where

$$c_g = \frac{\omega}{k}\frac{\alpha+3k^2}{2(\alpha+k^2)}, \quad p = \frac{\omega}{k^2}\frac{3k^4+6\alpha k^2-\alpha^2}{8(\alpha+k^2)^2},$$

$$q = -\omega k^2\frac{8\alpha^2+k^2\alpha+2k^4}{16(\alpha-2k^2)(\alpha+k^2)}, \quad r = -\frac{\omega}{k^3}\frac{(\alpha-k^2)(\alpha^2+6\alpha k^2+k^4)}{16(\alpha+k^2)^3},$$

$$u = \omega k\frac{(8\alpha^2+k^2\alpha+2k^4)(\alpha-k^2)}{32(\alpha-2k^2)(\alpha+k^2)^2}, \quad v = -3\omega k\frac{4k^8+4\alpha k^6-9\alpha^2k^4+\alpha^3k^2-8\alpha^4}{16(\alpha-2k^2)^2(\alpha+k^2)^2},$$

with

$$\omega^2 = k(\alpha+k^2), \quad \alpha = \frac{g\sigma}{\rho c^4}.$$

The potential $\overline{\phi}$ satisfies Laplace's equation

$$\overline{\phi}_{XX} + \overline{\phi}_{ZZ} = 0, \tag{2.3}$$

with boundary conditions

$$\overline{\phi}_Z = \frac{1}{2}\omega\frac{\partial}{\partial X}(|A|^2), \quad (Z=0), \quad \overline{\phi}_Z \to 0, \quad (Z\to-\infty). \tag{2.4}$$

Capillary-gravity solitary waves bifurcate when $\alpha = \alpha_0 = \frac{1}{4}$, $k = k_0 = \frac{1}{2}$, $\omega = \omega_0 = \frac{1}{2}$. The corresponding values of the coefficients are $p_0 = \frac{1}{2}$, $q_0 = \frac{11}{256}$, $r_0 = 0$, $u_0 = 0$, $v_0 = \frac{3}{32}$.

In terms of $\xi = X - c_g T$ and of $\tau = \epsilon T$, the evolution equations for A and $\overline{\phi}$ read

$$\mathrm{i}A_\tau + pA_{\xi\xi} + q|A|^2A + \mathrm{i}\epsilon(rA_{\xi\xi\xi} + uA^2A_\xi^* + v|A|^2A_\xi) - \epsilon kA\overline{\phi}_\xi|_{Z=0} = 0, \tag{2.5}$$

$$\overline{\phi}_{\xi\xi} + \overline{\phi}_{ZZ} = 0, \tag{2.6}$$

with boundary conditions

$$\overline{\phi}_Z = \frac{1}{2}\omega\frac{\partial}{\partial\xi}(|A|^2), \quad (Z=0), \quad \overline{\phi}_Z \to 0, \quad (Z\to-\infty). \tag{2.7}$$

Solutions in the form of envelope solitons can be sought as

$$A = R(\xi)\exp\{\mathrm{i}(\chi\tau + \epsilon f(\xi))\}\,, \tag{2.8}$$

which leads to the following system, correct to order ϵ^2:

$$2pR_\xi f_\xi + pRf_{\xi\xi} + rR_{\xi\xi\xi} + uR^2R_\xi + vR^2R_\xi \;=\; 0\,, \tag{2.9}$$
$$pR_{\xi\xi} - \chi R + qR^3 - \epsilon kR\overline{\phi}_\xi|_{Z=0} \;=\; 0\,. \tag{2.10}$$

Let us now expand R in powers of ϵ:

$$R = R_0(\xi) + \epsilon R_1(\xi) + \cdots\,.$$

One finds that R_0 satisfies the differential equation

$$pR_{0\xi\xi} - \chi R_0 + qR_0^3 = 0\,, \tag{2.11}$$

which gives

$$R_0 = \frac{a}{\cosh[a(\frac{q}{2p})^{1/2}\xi]}\,, \qquad \chi = \tfrac{1}{2}qa^2\,. \tag{2.12}$$

Now we explain the meaning of the small parameter ϵ. The branch of solitary waves bifurcates at $\alpha = \frac{1}{4}$. ϵ measures how far α is from α_0. Let $\mu = \alpha - \alpha_0$. One finds easily that

$$\mu = \frac{11}{256}a^2\epsilon^2\,.$$

Therefore

$$\epsilon R_0 = \frac{16}{\sqrt{11}}\sqrt{\mu}\frac{1}{\cosh[\sqrt{\mu}(x-t)]}\,.$$

The integration of the first equation of the system leads to

$$\epsilon f = -\frac{12}{11}\sqrt{\mu}\tanh[\sqrt{\mu}(x-t)]\,.$$

So far, we have only dealt with the central part of the envelope. Let us now compute the nonlocal term $\overline{\Phi}$:

$$\overline{\Phi} = \tfrac{1}{2}\,\mathrm{i}\,\omega\int_{-\infty}^{+\infty} \mathrm{e}^{iK\xi}\,\mathrm{sgn}\,K\,\widehat{R_0^2}\,\mathrm{d}K = -\tfrac{1}{2}\int_0^{+\infty}\sin(K\xi)\,\widehat{R_0^2}\,\mathrm{d}K\,.$$

One finds that

$$\overline{\Phi} = -\frac{64}{11}\int_0^\infty \frac{K\sin(K\xi)}{\sinh\left(\frac{8\pi}{a\sqrt{11}}K\right)}\,\mathrm{d}K\,.$$

This induced mean flow leads to a change $\overline{\eta}$ of the free surface elevation, given in unscaled variables by

$$\overline{\eta} = -\frac{1}{\alpha}\overline{\Phi}_t = -\frac{1}{22}\left(\frac{16}{\pi}\right)^3\mu^{3/2}\int_0^\infty K^2\,\frac{\cos[\frac{2K}{\pi}\sqrt{\mu}\,(x-t)]}{\sinh K}\,\mathrm{d}K\,.$$

The behavior as $\sqrt{\mu}(x-t)$ becomes large is given by

$$\overline{\eta} \sim \frac{512}{11\pi}\sqrt{\mu}\frac{1}{(x-t)^2}\,.$$

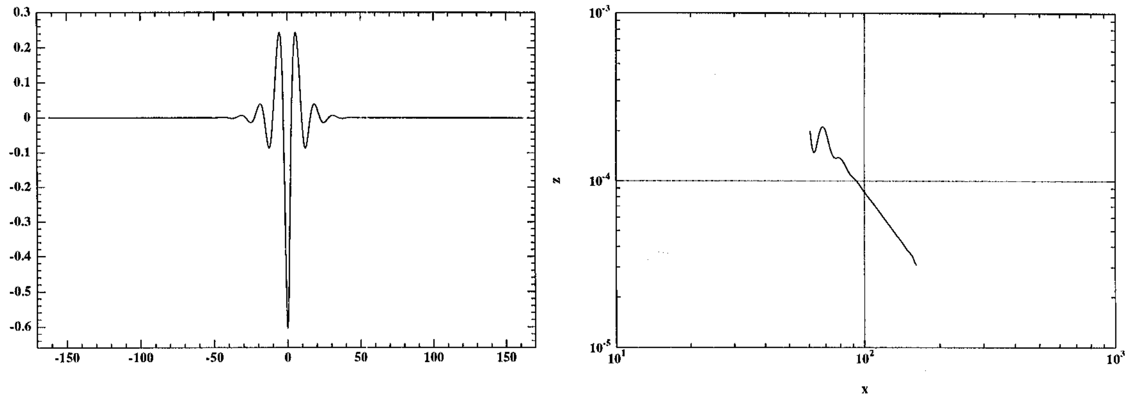

Figure 3.1: Solitary wave for $\mu = 0.02$. Profile and algebraic decay.

It follows that the algebraic tail dominates the exponential tail if

$$(x - t) > -\frac{\ln \mu}{\sqrt{\mu}}.$$

For small amplitude waves, when the algebraic tails starts to dominate, it is already quite small and therefore one can conclude that the effect of the algebraic tail is more pronounced for larger amplitude waves. In the next section, numerical results on the full Euler equations are used to show that it is indeed the case.

3 Numerical results

In this section, we present numerical results for several values of μ. For large values of μ, it is clear that the tail decays algebraically (see plot of $\ln \eta$ versus $\ln x$). The numerical solutions are obtained by using the scheme of Vanden–Broeck & Dias (1992).

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