A shortcut for computing time-domain free-surface potentials avoiding Green function evaluations.

A. CLÉMENT

ÉCOLE CENTRALE de NANTES
Laboratoire de Mécanique des Fluides - division Hydrodynamique Navale
CNRS URA 1217, E.C.N, Nantes, FRANCE

The numerical solution of transient hydrodynamic problems in the frame of the linearized potential theory requires the computation of convolution integrals. These integrals may be regarded as the memory of the free-surface fluid. Since they extend from the initial state of rest up to the current time *t*, the mass storage and *cpu* time required for their computation grow quickly with time, roughly quadratically. Consequently, in time-domain seekeeping computations, the major part of *cpu* is spent in evaluating these convolutions (Magee 1991).
Let us consider, for instance, the generation of surface waves by the prescribed motion ($\mathbf{V}$) of a body around its equilibrium position (S) in a perfect fluid.
The resulting velocity potential $\Phi(M,t)$ must satisfy the following boundary integral equation :

$$\frac{\Phi(M,t)}{2}-\iint_S \Phi(M',t)\frac{\partial G_0(M,M')}{\partial n'}ds' = -\iint_S \mathbf{V}.\mathbf{n}(M',t)G_0(M,M')ds'$$
$$+\iint_S ds'\int_0^t \Phi(M',\tau)\frac{\partial F(M,M',t-\tau)}{\partial n'}d\tau - \iint_S ds'\int_0^t \mathbf{V}.\mathbf{n}(M',t)F(M,M',t-\tau)d\tau \qquad (1)$$

where G_0 and F are respectively the impulsive and the memory part of the Green function. In the present study we focus our attention on the convolution integrals in the RHS of (1). They may be written in the general form :

$$S=\int_0^T Q(\tau)F(r,Z+Z',t-\tau)d\tau \qquad (2)$$

where, when the water depth is infinite (Finkelstein 1957, Wehausen & Laitone 1960) :

$$F(r,\zeta,\xi)=\int_0^\infty \sqrt{K}\sin(\sqrt{K}\xi)\mathrm{J}_0(Kr)\mathrm{e}^{K\zeta}\,dK \qquad (3)$$

with : $r=\sqrt{(X-X')^2+(Y-Y')^2}$

Up to now, the efforts made to speed up the numerical computation of integrals like (2) in the numerical implementations of BEM to solve integral equations like (1) were essentially :

- derivation of alternative faster expressions of the Green function F, better suited to numerical calculation (Jami (1982), Newman (1985), Beck & Liapis (1987), ...),
- tabulation of the memory part of the Green function in order to replace the evaluation of F by a bi-linear interpolation in

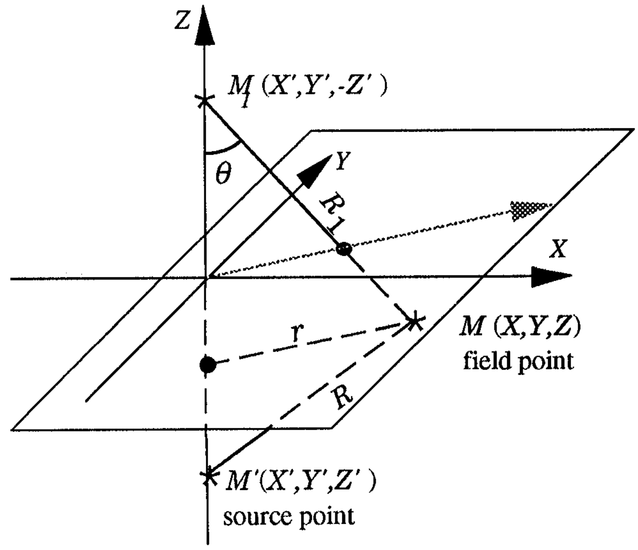

Fig.1 : Definition sketch.

a precomputed table (Ferrant-1988, Magee & Beck-1989).
An alternative method to evaluate the convolution products (2) without computing explicitly the Green function was proposed by Clément (1991). It is based on the identification of the Green function considered as a SISO (Single-Input-Single-Output) linear time-invariant process. The identification parametric *model* of the process is a linear ODE linking the input $Q(M',t)$, the output $S(M,t)$, and their derivatives. Once such a model has been found, S can be recovered from the knowledge of Q by simply integrating the ODE from a time step to the next one, instead of computing convolution integrals like (2). Doing so could save a huge amount of computer time and memory (Clément 1992).

1. A parametric time-varying model

In our first papers related to this topic (1991-1992), we attempted to identify the Green function with discrete time invariant models. These kind of models, often called ARX in process science literature, are characterized by discretized ODEs with constant coefficient.
They were shown later (Clément 1995) to be inadequate for the time domain Green function (3) which behaves asymptotically like a *"chirp"* process. This feature results in increasing considerably the model order to maintain a reasonable accuracy as both source point and field points approach the free surface (*i.e* : $\mu \to 0$).
Thus, we were led to adopt a more refined model (4) where the ODE coefficients are themselves function of time.

$$\sum_{i=0}^{i=n} A_i(t) S^{(i)}(t) = \sum_{i=0}^{i=n-1} P_i(t) Q^{(i)}(t) \tag{4}$$

where we use the notation : $S^{(i)}(t) = \dfrac{d^i S(t)}{dt^i}$.
In such a differential model, the causality of the process ensures the right-hand side order to be less than the left-hand side order. This property which is well known when the coefficients are constant, still holds for time-varying models (Zadeh et al. 1963).

2. The auto-regressive terms $A_i(t)$

The left-hand side of (4) is generally referred to as the *auto-regressive* part of the model. It can be obtained from the response of the process to an impulsive input $Q(t) = \delta(t)$.
Here, the impulse response function of the process is, by definition, the Green function itself $F(M,M',t)$.
Taking advantage of the fact that the kernel of the integral (3) can be expressed by an hypergeometric function, an exact fourth order differential equation for F may be derived making use of the general confluent equation :

$$R_1^2 \frac{\partial^4 F}{\partial t^4} + \mu R_1 t \frac{\partial^3 F}{\partial t^3} + \left(\frac{t^2}{4} + 4\mu R_1 \right) \frac{\partial^2 F}{\partial t^2} + \frac{7t}{4} \frac{\partial F}{\partial t} + \frac{9}{4} F = 0 \tag{5}$$

with : $\mu = -(Z+Z') / \sqrt{r^2 + (Z+Z')^2}$. Thus, from (5), the auto-regressive coefficients $A_i(t)$ are found to be polynomials at most of degree two in the time variable. Their coefficients are very simple functions of the geometric parameters μ and R_1. A detailed derivation of (5) will appear in a more lengthy paper (Clément 1997).
As a first step toward computations speed up, one may use this ODE instead of the classical series developments (Newman 1985) for the *in-line* evaluation of the Green function in the numerical computation of the integral (3). To do so, one need also the initial conditions which can be easily deduced from (3) and its time derivatives by using the integral form of Legendre polynomials. After some algebra, we obtain :

$$\begin{cases} F^{(2k)}(r,\zeta,0)=0 \\ F^{(2k+1)}(r,\zeta,0)=\dfrac{(-1)^k(k+1)!}{R_1^{k+2}}\mathrm{P}_{k+1}(\mu) \end{cases} \qquad k=0,1,\ldots \tag{6}$$

which give the complete set of the time derivatives at the origin ; all the even order derivatives are null. It should be noticed to conclude this section that (5) and (6) are exact analytical results.

3. The forcing terms

The right-hand side of eq.(4) is generally referred to as the forcing term. Its form is a priori unknown, and a direct combination of (2) and (5) would lead to reintroduce convolution integrals in the RHS of the model. Thus, referring to time invariant models for which the property is formally established, we made the hypothesis that the forcing term of the present model can be expressed by a differential form similar to LHS (*i.e* with polynomial coefficients), and we sought it in the form :

$$RHS(4)=\sum_{i=0}^{3}Q^{(i)}(t)\sum_{j=0}^{j=\beta}p_{ij}t^j \tag{7}$$

The determination of the unknown coefficients p_{ij} was made easy by the knowledge of all the Markov parameters of the process through eq.(6). The method consists in expressing the model (4) and its successive time derivatives at the origin of time. At each level of differentiation, one can show that the lowest order unknown parameters p_{ij} may be expressed as a linear combination of the coefficients of (5), and of the Markov parameters $F^{(j)}(.,.,0)$.

4. continuous models of the Green function and its gradient.

The above method was applied first to the Green function itself , and gave :

$$R_1^2S^{(4)}+\mu R_1tS^{(3)}+\left(\frac{t^2}{4}+4\mu R_1\right)S^{(2)}+\frac{7t}{4}S^{(1)}+\frac{9}{4}S=\mu Q^{(2)}+\frac{\mu^2t}{R_1}Q^{(1)}+\left[\frac{\mu\left(-3\mu^2+\frac{13}{4}\right)}{2R_1^2}t^2+\frac{1+\mu^2}{R_1}\right]Q \tag{8}$$

The maximum order of the polynomials P_i was assumed to be at most equal to the order of the A_i to ensure a stable asymptotic behaviour; whatever the input of the process ; nevertheless, it should be pointed out that the iterative method in §3 could provide higher order polynomials.

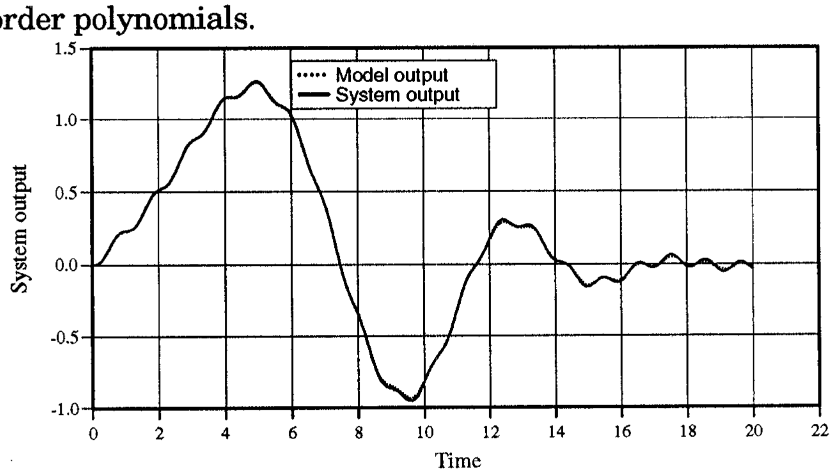

Fig.2 : Output $S(t)$ for input $Q(t)=\sin(6t)$, computed by both methods. $\mu=.3714$

Results of a simulation of the process output for an harmonic input $Q(t)=\sin(6t)$ are plotted on Fig.2 . Both methods were applied : a standard trapezoidal integration method using (2) and (3), and the present time-varying model (8). Discrepancies between these two curves appear to remain negligible in this precise case.

Because the solution of the integral equations of time-domain hydrodynamics may require also convolution integrals involving the gradient of the Green function, the present approach was applied to the horizontal and vertical gradient as well.
We simply give below the results after calculations.

Horizontal gradient

$$R_1^2\frac{\partial^4 S}{\partial t^4}+\mu R_1 t\frac{\partial^3 S}{\partial t^3}+\left(\frac{t^2}{4}+6\mu R_1\right)\frac{\partial^2 S}{\partial t^2}+\frac{11t}{4}\frac{\partial S}{\partial t}+\frac{21}{4}S=\frac{-3\sqrt{1-\mu^2}}{R_1}\left\{\mu Q^{(2)}+\frac{\mu^2 t}{R_1}Q^{(1)}+\left[\frac{\mu\left(-5\mu^2+\frac{17}{4}\right)}{2R_1^2}t^2+\frac{1+\mu^2}{R_1}\right]Q\right\} \tag{9}$$

Vertical gradient

$$R_1^2\frac{\partial^4 S}{\partial t^4}+\mu R_1 t\frac{\partial^3 S}{\partial t^3}+\left(\frac{t^2}{4}+6\mu R_1\right)\frac{\partial^2 S}{\partial t^2}+\frac{11t}{4}\frac{\partial S}{\partial t}+\frac{25}{4}S= \tag{10}$$

$$\left(\frac{3\mu^2-1}{R_1}\right)Q^{(2)}+\frac{\mu t}{R_1}\left(\frac{3\mu^2-1}{R_1}\right)Q^{(1)}+\left[\left(-15\mu^4+\frac{75\mu^2}{4}-\frac{13}{4}\right)\frac{t^2}{2R_1^3}+3\mu\frac{\left(\mu^2+1\right)}{R_1^2}\right]Q$$

Numerical simulations were also performed for these two models with the same input, and a comparable accuracy was observed. Thus, the proposed models seem to be useful for our purpose in that frequency range. Nevertheless, the results are not so good as the input frequency decreases, and refining the models of the forcing terms seems to be necessary in that range.

Conclusion The time-domain Green function and its gradient were found to be solutions of fourth order ordinary differential equations with time-varying coefficients. These coefficients functions are low order polynomials of the time variable, and their own coefficients are simple functions of the geometrical parameters of the problem.
These time varying models may be used to compute the convolution integrals in time-domain seakeeping codes without computing the Green function itself (nor its gradient).
The accuracy is excellent for high frequency, but remain to be improved in the low frequency range.

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DISCUSSION

Newman J.N.: Equation 5 is remarkable. One consequence is that the corresponding frequency-domain Green function integral satisfies a 2nd order ODE with respect to the wavenumber!

Magee A.: I wish to congratulate the author on a truly original contribution on the use of time-domain Green functions. If a more accurate method for the forcing terms can be found, this shortcut should soon supplant all available methods for computing the time-domain Green function, because it will permit a gain of about 80 % in memory requirements for typical calculations. It is clear that equation (5) is exact. Is it possible to find exact solutions for the forcing terms (second part of eqn. 8)? What are the steps necessary to find these terms?
Secondly, you have treated the case applicable to linearised motions at zero forward speed, that is, the positions of the source and field points are not functions of time. However, we already have well-developed frequency-domain calculation methods for this case, at least in infinite depth. The real benefit of the time-domain method is its applicability to more complex cases such as steady forward speed and arbitrary large-amplitude motions because the Green function retains its relatively simple form in these cases as well.
According to my calculations, equation (5) is also valid in a steadily moving coordinate system (linearised problem with steady forward speed U) provided we replace the partial time derivative $\frac{\partial}{\partial t}$ with the total derivative $\frac{\partial}{\partial t} - U\frac{\partial}{\partial x}$ in the steadily moving frame. In this case we would have:

$$\underbrace{R_1^2\left(\frac{\partial}{\partial t} - U\frac{\partial}{\partial x}\right)^4 F}_{(1)} + \underbrace{\mu R_1 t\left(\frac{\partial}{\partial t} - U\frac{\partial}{\partial x}\right)^3 F}_{(2)} + \underbrace{\left(\frac{t^2}{4} + 4\mu R_1\right)\left(\frac{\partial}{\partial t} - U\frac{\partial}{\partial x}\right)^2 F}_{(3)} + \underbrace{\frac{7t}{4}\left(\frac{\partial}{\partial t} - U\frac{\partial}{\partial x}\right)F}_{(4)} + \underbrace{\frac{9}{4}F}_{(5)} = 0 \quad (5bis)$$

where R_1 and μ are functions of time. In order to calculate the Green function in this case, we would need to "simulate" the Green function and its first four x-derivatives, which I have not done here. However, I have tested (5bis) by other means.
The attached figure presents the five terms of equation (5bis) and the sum of the terms, which should equal zero, if the relation holds. The Green function values were calculated using a Romberg method to assure a good precision, and the

derivatives were calculated using finite-difference schemes. The results seem to indicate that the relation holds and this is true for all values of the parameters tested. This calculation is confirmed by an analytical calculation (Maple) using the series expansion of the Green function (up to the order of the truncated series). Furthermore, I believe the same equation should generalise to the case of arbitrary motion of the source and field points (large-amplitude motions case) by using $\frac{\partial}{\partial t} - \vec{V}.\nabla$, where $\vec{V}$ is the relative velocity between M and M' in place of the partial time derivative $\frac{\partial}{\partial t}$ of equation (5). If this is true, then the large-amplitude calculations would be only slightly more time-consuming than linearised ones–a great advance indeed! Could you please comment?

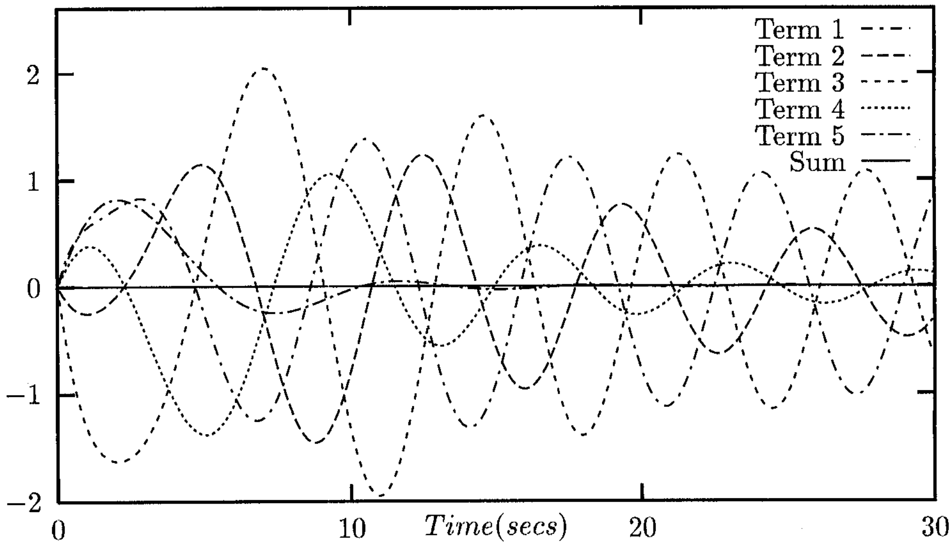

Fig. 1. *The 5 terms of eqn (5bis) and their sum, in the case of steady forward speed* $U = 0.25$, $x = 1$, $y = 0$, $z = -2$

Clément A.: It is indeed possible to find exact solutions for the forcing term. The simplest method consists in differentiating (2) four times using the Leibnitz rule, and then integrating (5) after having multiplied it by $Q(t)$. After a few lines of calculations, the exact forcing term of (4) is obtained. Unfortunately, it contains new convolution integrals, which is exactly what we want to avoid in our model! Thus, we chose the present approximation by a differential form, with no guarantee of convergence.

The Green function F does not depend on the trajectory of the source, due to the impulsive nature of its strength, and then (5) is also valid in that case, expressed in a fixed reference frame, provided R_1 and μ are understood as $R_1(0)$ and $\mu(0)$.
It can be indeed expressed in a moving reference frame by changing the derivative operator as you did, and taking into account the induced dependence of the space parameters on time. Thus, your numerical check of the ODE in these conditions is not surprising. As you mention, it involves higher horizontal derivatives of F. Differential equation similar to (5) could be easily derived for them, from the general lemma established in Clément (1997), to appear shortly.