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Dispersion relation and far-field waves

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A theoretical formulation of wave diffraction-radiation by ships or offshore structures, motivated by the practical and theoretical importance of free-surface potential flows and the formidable complexities of existing calculation method based on free-surface Green function, is recently developed and summarized in *Noblesse, Chen and Yang* (1996). One of important results is the analysis of the classical Fourier representation of free-surface effects, as a two-dimensional linear superposition of elementary plane progressive waves $\exp[-i(\alpha\xi+\beta\eta+ft)]$, given in *Noblesse and Chen* (1995), which defines the wave potential $\phi^W(\xi,\eta)$ in terms of the single Fourier integral

$$4\pi\phi^W = -i\sum\int_{D=0} ds[\text{sign}(D_f)+\text{sign}(\xi D_\alpha+\eta D_\beta)]\exp[-i(\xi\alpha+\eta\beta)]A/\|\nabla D\| \tag{1}$$

along every curve, called dispersion curve, defined in the Fourier plane (α,β) by the dispersion relation $D=0$. Here, ds is the arc length along a dispersion curve, $\|\nabla D\|^2 = D_\alpha^2+D_\beta^2$ and f is the frequency. The Fourier representation (1) is valid for steady and time-harmonic free-surface flows, in infinite or finite water depth, generated by an arbitrary distribution of singularities defined by the generic amplitude function A, which is given by a distribution of the elementary wave function $\exp[kz+i(\alpha x+\beta y)]$ over the surface of the wave generator (e.g. ship or offshore structure). Here, $k=\sqrt{\alpha^2+\beta^2}$ is the wavenumber.

Considerable information about important far-field features of the waves defined by the Fourier representation (1) have been revealed in *Chen* (1996), via a stationary-phase analysis of (1). Specifically, the constant-phase curves (e.g. crest lines) and the related wavelengths, directions of wave propagation, and phase and group velocities can be determined explicitly from the dispersion function D. This stationary-phase analysis of (1), which provides direct relationships between the dispersion curves $D=0$ in the Fourier plane and the corresponding wave systems in the physical plane, is briefly summarized here for the generic case of dispersive waves characterized by an arbitrary dispersion function D, and for the specific case of time-harmonic ship waves in deep water.

Generic dispersive waves

The far-field features of ϕ^W are determined by the stationary points of the phase function $\varphi=\xi\alpha+\eta\beta$ along the dispersion curves. The stationary points are defined by $\varphi'=\xi\alpha'+\eta\beta'=0$ and satisfy the relation :

$$\xi D_\beta - \eta D_\alpha = 0 = h\|\nabla D\|\sin(\gamma-\theta) \tag{2}$$

Here, h and θ are the polar coordinates of the field point $(\xi,\eta)=h(\cos\theta,\sin\theta)$. Furthermore, γ is defined by $(\cos\gamma,\sin\gamma)=(D_\alpha,D_\beta)/\|\nabla D\|$ and thus represents the angle between the unit vector normal to a dispersion curve and the α axis. The wavelength of the waves corresponding to a stationary point (2) is given by $\lambda=2\pi/k$ where k is the wavenumber at the stationary point.

Expression (2) shows that a point of stationary phase on a given dispersion curve is defined by $\gamma=\theta$ or $\gamma=\theta+\pi$. Thus, a point of a dispersion curve generates waves in the physical space in a direction normal to the dispersion curve. The sign function $\text{sign}(\xi D_\alpha+\eta D_\beta)$ in (1) is equal to 1 if $\gamma=\theta$ or -1 if $\gamma=\theta+\pi$. Expression (1) therefore indicates that a point of a dispersion curve generates waves in the direction of the normal vector ∇D to the dispersion curve if $\text{sign}(D_f)=1$, or in the opposite direction if $\text{sign}(D_f)=-1$. Furthermore, at the stationary point $\varphi'=0$, the second derivative of the phase function is expressed as :

$$\varphi'' = c\sqrt{\alpha'^2+\beta'^2}\,d \quad \text{with} \quad d=h(\xi\alpha'-\eta\beta')/(2\xi\eta) \tag{3}$$

where α' and β' are differentiation of α and β with respect to the integral variable along the dispersion curves, and the curvature c is given by :

$$c=(-D_\alpha^2 D_{\beta\beta}+2D_\alpha D_\beta D_{\alpha\beta}-D_\beta^2 D_{\alpha\alpha})/\|\nabla D\|^3 \tag{4}$$

As $d\neq 0$ in the expression (3), $\varphi''=0$ only at the point of inflection where $c=0$. Two points on both sides of the inflection point may have the same unit normal and then two groups of waves may propagate in the same direction but with different wave number. In fact, an inflection point (α_c,β_c) of a dispersion curve, determined by $c = 0$, defines a cusp line along which two distinct wave systems are found. The corresponding angle γ_c is defined by

$$\gamma_c = \tan^{-1}(D_\beta / D_\alpha)_c \tag{5}$$

where the subscript c indicates evaluation at $(\alpha_c\,, \beta_c)$.

The curves along which the phase φ is constant, equal to $C_n^{\pm} = \pm 2\pi - \text{sign}(\varphi'')\pi/4$, are given by

$$(\xi, \eta) = C_n^{\pm}(D_\alpha, D_\beta)/(\alpha D_\alpha + \beta D_\beta) \quad \text{with} \quad \text{sign}(C_n^{\pm}) = \text{sign}(\alpha D_\alpha + \beta D_\beta)\,\text{sign}(D_f) \tag{6}$$

The phase velocity $\vec{v}^f$, determined by the stationary-phase relation (2), is given by

$$\vec{v}^f = -(\alpha, \beta) f / k^2 \tag{7}$$

which is orthogonal to constant-phase curves (6) and different, both in magnitude and in direction, from the group velocity $\vec{v}^g$, at which wave energy is transported, defined by

$$\vec{v}^g = -(\partial f/\partial\alpha, \partial f/\partial\beta) = (D_\alpha, D_\beta)/D_f \tag{8}$$

Expresions (8) and (6) yield $(\xi\,, \eta) \cdot \vec{v}^g > 0$, which shows that wave energy is propagated away from a wave generator in accordance with the radiation condition.

Far-field features of time-harmonic ship waves

The foregoing results, valid for generic dispersive waves, are now applied to the particular case of time-harmonic ship waves in deep water, for which the dispersion function is given by

$$D = (f - F\alpha)^2 - k \tag{9}$$

For $\tau = fF < 1/4$, three dispersion curves defined by $D = 0$ intersect the axis $\beta = 0$ at four values of α, denoted $\alpha_o^{\pm}$ and $\alpha_i^{\pm}$. The *ring, inner V* and *outer V* waves correspond to the interior curve comprised between α_i^- and α_i^+, the exterior right curve located in $\alpha_o^+ \le \alpha < \infty$, and the exterior left curve located in $-\infty < \alpha \le \alpha_o^-$, respectively. For $\tau > 1/4$, only two distinct dispersion curves intersect the axis $\beta = 0$ at α_i^+ and α_o^+. The *ring-fan* and *inner V* waves are respectively associated with the dispersion curves in the left $(-\infty < \alpha \le \alpha_i^+)$ and right $(\alpha_o^+ \le \alpha < \infty)$ regions.

The wavelengths of the *transverse* waves (the waves at the ship track $\eta = 0$), in the various component wave systems described above, have already been given in *Noblesse, Chen and Yang* (1996). In th same way, the wavelengths at the edges (cusp lines) of the wedges containing the inner and outer V waves and the ring-fan waves are given by $\lambda_c = 2\pi/k_c$ where k_c is the wavenumber at the inflection points determined by the relation

$$F^4 k_c^2 - (3/2)F^2 k_c + \text{sign}(f - F\alpha) 4\tau F\sqrt{k_c} - 3\tau^2 = 0 \tag{10}$$

The corresponding wedge angle γ_c is

$$\gamma_c = \tan^{-1}(\pm 1/\sqrt{6F^2 k_c - 1}\,) \tag{11}$$

The group velocity (8) is now written as

$$\vec{v}^g = -[F + \text{sign}(f - F\alpha)\alpha/(2k^{3/2}), \text{sign}(f - F\alpha)\beta/(2k^{3/2})] \tag{12}$$

in the system of coordinates moving with the mean forward motion of the ship, and

$$\vec{V}^g = \vec{v}^g + (F, 0) = -\,\text{sign}(f - F\alpha)(\alpha, \beta)/(2k^{3/2}) \tag{13}$$

in the absolute system of coordinates. The absolute velocity $\vec{V}^g$ is orthogonal to the constant-phase curves, whereas the relative velocity $\vec{v}^g$ is not.

The foregoing simple analytical relationships between the dispersion curves in the Fourier plane and important features of the corresponding far-field waves in the physical plane are illustrated in the attached figures for the four distinct cases which must be considered for time-harmonic flows with forward speed.

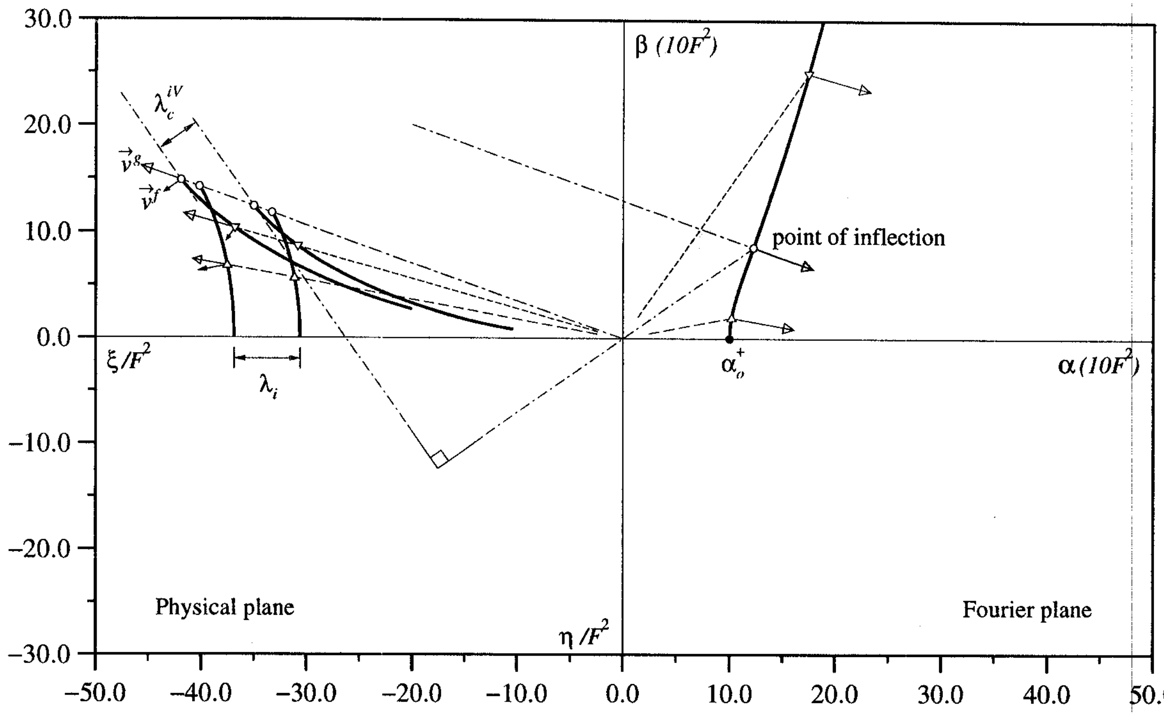
References

F. Noblesse, X.B. Chen and **C. Yang** (1996) *Fourier-Kochin theory of free-surface flows*, 21st Symposium on Naval Hydrodynamics, Trondheim, Norway.

F. Noblesse and **X.B. Chen** (1995) *Decomposition of free-surface effects into wave and near-field components*, Ship Technology Research (42) 167-185.

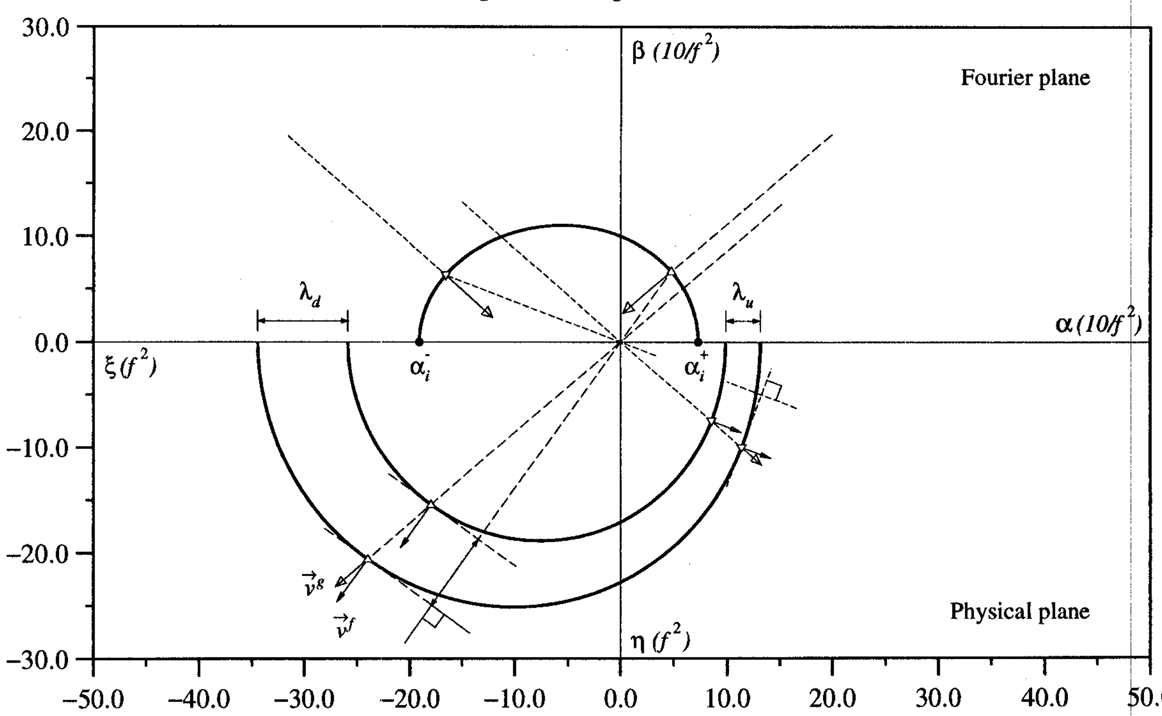
X.B. Chen (1996) *Evaluation des champs de vagues générés par un navire avançant dans la houle*, Rapport final du Projet DRET/BV no.95 378.

Figure 1: Inner V waves



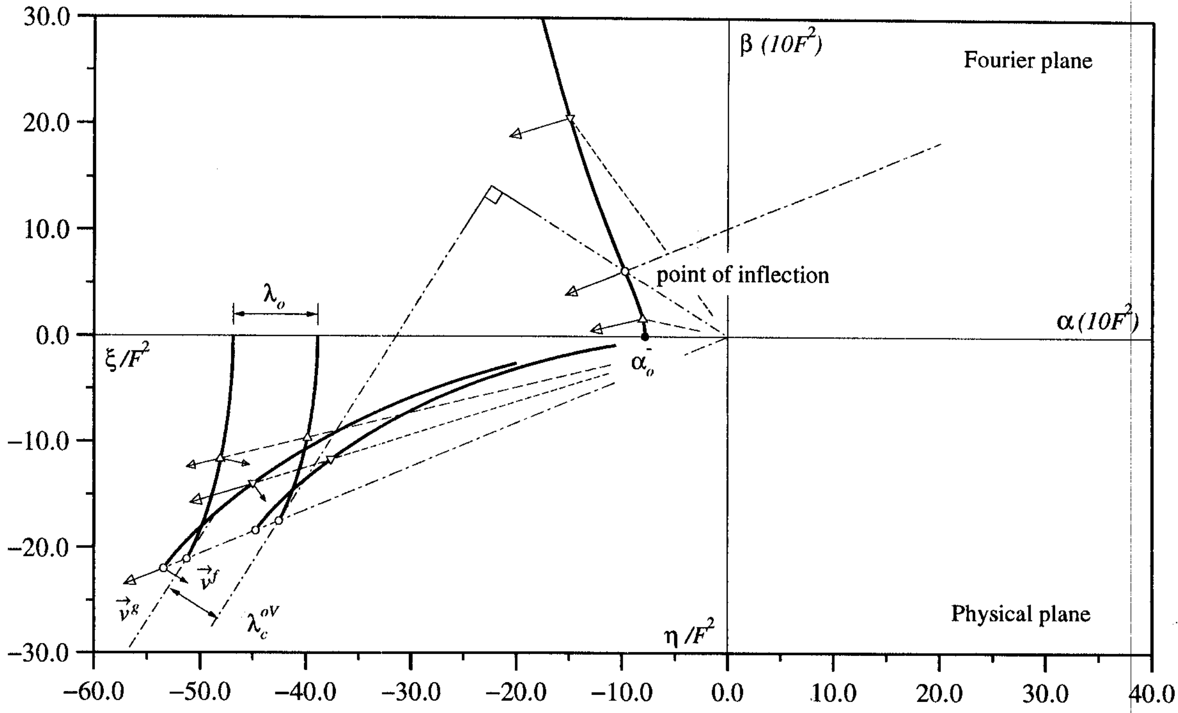
The right exterior dispersion curve ($\alpha_o^+ \leq \alpha < \infty$) is associated with the inner V waves, for $\tau \geq 0$. Two groups of waves systems (the transverse and divergent waves) correspond to two portions of the dispersion curve ($\alpha_o^+ \leq k \leq k_c$) and ($k_c \leq k < \infty$), respectively.

Figure 2: Ring waves



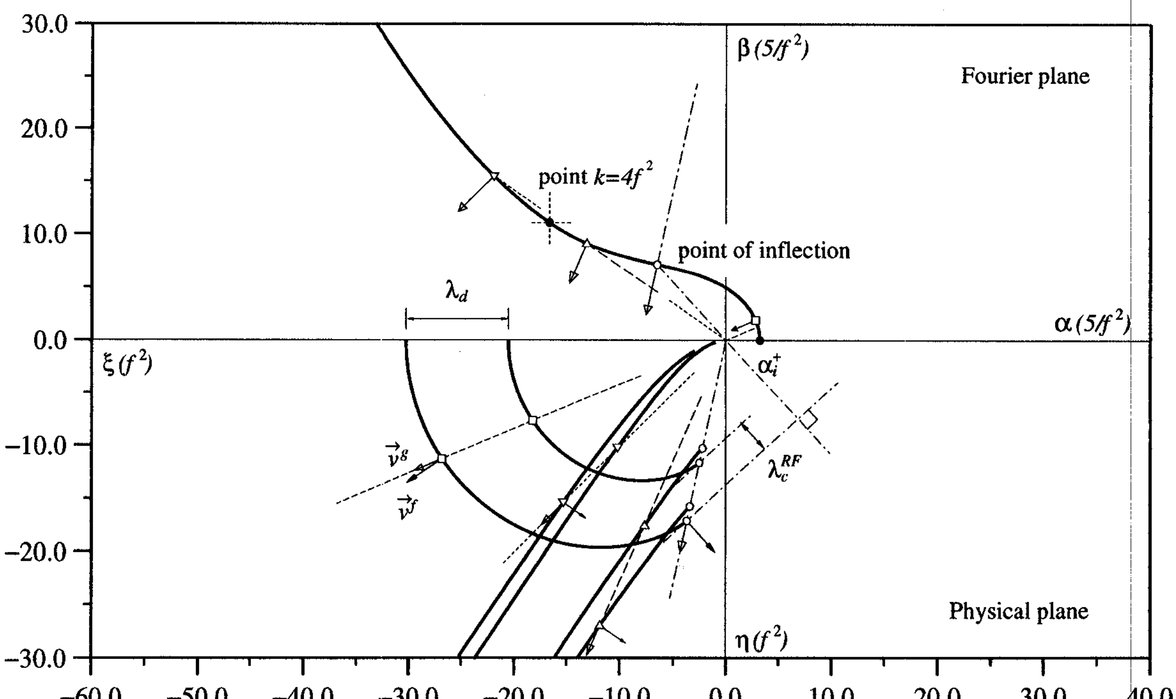
The interior dispersion curve comprised between α_i^- and α_i^+ is associated with the ring waves, for $\tau < 1/4$.

Figure 3: Outer V waves



The left exterior dispersion curve ($-\infty < \alpha \leq \alpha_o^-$) is associated with the outer V waves, for $\tau < 1/4$. Two groups of waves systems (the transverse and divergent waves) correspond to two portions of the dispersion curve ($-\alpha_o^- \leq k \leq k_c$) and ($k_c \leq k < \infty$), respectively.

Figure 4: Ring-fan waves



The left dispersion curve ($-\infty < \alpha \leq \alpha_i^+$) is associated with the ring-fan waves, for $\tau > 1/4$. Three groups of waves systems (the partial-ring waves, the outer-fan waves and the inner-fan waves) correspond to three portions of the dispersion curve ($\alpha_i^+ \leq k \leq k_c$), ($k_c \leq k \leq 4f^2$) and ($4f^2 \leq k < \infty$), respectively.

DISCUSSION

Schultz W.W.: What new conclusions (or discrepancies) are obtained in your Fourier analysis over the simple ray theory of Eggers (1957)?

Chen X.B., Noblesse F.: The results and the analysis we have summarized differ from those given in Eggers (1957) and elsewhere, in a number of ways. First of all, our results are valid for generic dispersive waves generated by arbitrary distributions of singularities. Thus, the results can directly be applied to a broad class of dispersive waves, including steady and time-harmonic water waves with or without forward speed in homogeneous or density-stratified water of infinite or finite depth. The results we have given provide simple and elegant explicit relationships between the so-called dispersion curves, defined in the Fourier plane by the dispersion relation and the corresponding far-field waves. These relationships include expressions, both in fixed (attached to the earth) and moving (attached to a translating distribution of singularities) systems of coordinates, for the phase and group velocities of the various wave components associated with each distinct dispersion curve. It is also shown that cusp lines of far-field wave patterns are explicitly related to inflection points of the dispersion curves, which yield closed-form expressions for cusp-angles. In particular, for the case of time-harmonic ship waves in deep water considered for illustrative purposes, two particular *exact* values of τ, namely $\tau = \sqrt{2/27}$ (at which no waves propagate upstream) and $\tau = \sqrt{8/3}$ (where unsteady waves are contained within the wedges of the steady waves), are given (to the authors' knowledge, only numerical approximations to these exact values of τ have previously been given).

Magee A.: Using the relation you developed for group velocity, for a given τ and F, can you calculate the time for a disturbance to reflect off tank walls and return to the ship. In other words, can you find the τ and F values free from tank reflections?

Chen X.B., Noblesse F.: Indeed, the relationship we have given, specifically the expressions for the wave propagation angles and the group velocity, can be directly used to determine the time required for the various components of the waves diffracted-radiated by a ship model advancing at constant speed in a water tank to be reflected at the walls of the tank.