# On the Wave Resistance of a 2D Body in a Two-Layer Fluid

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## 1. Introduction

A cylindrical body, moving forward with constant velocity in an inviscid, incompressible fluid under gravity, is considered. The fluid consists of two layers having different densities, and the body is totally submerged either in the upper layer or in the lower one (see fig. 1.1 and 1.2 respectively). The resulting fluid motion is assumed to be steady state in the coordinate system attached to the body. In the framework of the linearized water wave theory the corresponding boundary value problem is usually referred to as the Neumann–Kelvin problem for a two-layer fluid.

The first description of the dead water phenomenon, occurring in a two-layer fluid, was published more than 90 years ago [1]. Further results one can find in the survey paper [2], recent development is covered by [3] (see also references cited therein). However, the existing papers are devoted either to numerical computations or to investigation of the fundamental solutions.

Here we announce the theorem of unique solvability, which have not been obtained for the problem to the present day. Also, a formula is derived, which expresses the wave resistance in terms of the velocity potential asymptotics at infinity downstream. Another formula for the wave resistance connects it with energy transportation by two modes of waves.

## 2. Statement of the problem and the solvability theorem

The geometrical notations are given in figure 1.

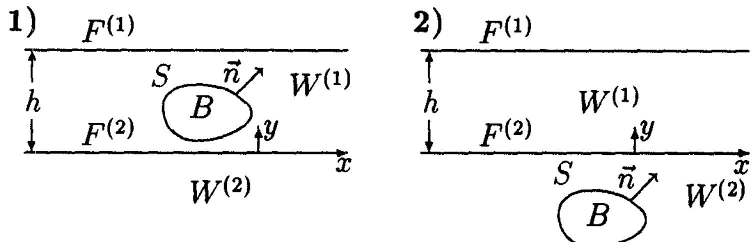


**Figure. 1**

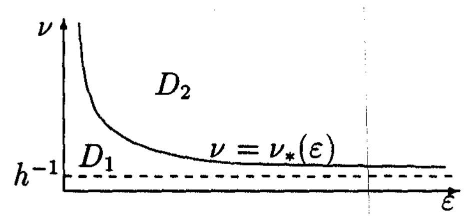


**Figure. 2**

The upper layer of the density $\rho_1$ is bounded from above by the free surface $F^{(1)} = \{y = h\}$. The lower layer of the density $\rho_2 > \rho_1$ has the infinite depth, and the coordinate origin is placed in the interface $F^{(2)}$.

The fluid motion in the upper (lower) layer is described by the velocity potential $u^{(1)}$ ($u^{(2)}$). The potentials must satisfy the boundary value problem:

$$\nabla^2 u^{(i)} = 0 \quad \text{in} \quad W^{(i)}, \quad i = 1,2, \quad u^{(1)}_{xx} + \nu u^{(1)}_y = 0 \quad \text{on} \quad F^{(1)}, \tag{1}$$

$$u^{(1)}_y = u^{(2)}_y, \quad \rho_1[u^{(1)}_{xx} + \nu u^{(1)}_y] = \rho_2[u^{(2)}_{xx} + \nu u^{(2)}_y] \quad \text{on} \quad F^{(2)}, \tag{2}$$

$$\partial u^{(i)}/\partial n = U\cos(n,x) \text{ on } S, \text{ bounding } W^{(i)} \text{ internally}, \tag{3}$$

$$\sup_{W^{(i)}} |\nabla u^{(i)}| < \infty, \quad \lim_{x\to+\infty} |\nabla u^{(i)}| = 0, \quad i = 1,2. \tag{4}$$

Here $U$ is the constant speed of the cylinder, $\nu = g\,U^{-2}$, and $g$ is the acceleration due to gravity.

It is well-known from the results on sources, that there exist two regimes of flow about the body. Namely, if $\nu > \nu_* = (1+\varepsilon)/\varepsilon h$, where $\varepsilon = \rho_2/\rho_1 - 1$ (see domain $D_2$ in fig. 2), then there exist both surface and internal waves behind the body. Their wavenumbers are $\nu$ and $\nu_0$ respectively, where $\nu_0$ is the only positive root of the equation $Q(\nu_0) = 0$ existing in this case. Here $Q(k) = (1+\varepsilon)k + (k - \varepsilon\nu)\tanh(kh)$. If $\nu < \nu_*$ (domain $D_1$), then there are only surface waves at infinity downstream.

Using methods of the potential theory in the same way as in [4, 5] the following theorem can be proved.

**Theorem.** *The problem (1)–(4) has the unique solution for all pairs $(\varepsilon, \nu) \in D_1$ $(D_2)$, with possible exception for a set, which is dense nowhere in $D_1$ $(D_2)$.*

To give an idea of a possible exceptional set we mention that every line $\nu = \text{const} \neq h^{-1}$ (with possible exception for a finite number of lines) can contain no more than a discrete sequence of exceptional points. When $\nu > h^{-1}$, then the only limit point of the sequence is on the curve dividing $D_1$ and $D_2$. And if $\nu < h^{-1}$, then such a line contains only a finite number of possible exceptional points.

## 3. Asymptotics at infinity

Derivation of the formula for wave resistance in the next section is based upon the asymptotic representation of the velocity potentials satisfying (1)–(4) at infinity. For the given right hand side in the Neumann condition (3) we have as $|z| \to \infty$ $(z = x + iy)$:

$$u^{(1)}(z) = C_\pm + \psi_\pm^{(1)}(z) + H(-x)\Big\{(\mathcal{A}\sin\nu x + \mathcal{B}\cos\nu x)e^{\nu y}$$
$$+ H(\nu - \nu_*)\Big[\Big(1 + \varepsilon - \varepsilon\nu\nu_0^{-1}\Big)\cosh\nu_0 y + \sinh\nu_0 y\Big](\mathcal{A}_0 \sin\nu_0 x + \mathcal{B}_0\cos\nu_0 x)\Big\}, \quad \pm x > 0 \tag{5}$$

$$u^{(2)}(z) = C_2 + \psi_\pm^{(2)}(z) + H(-x)\{(\mathcal{A}\sin\nu x + \mathcal{B}\cos\nu x)e^{\nu y}$$
$$+ H(\nu - \nu_*)(\mathcal{A}_0\sin\nu_0 x + \mathcal{B}_0\cos\nu_0 x)e^{\nu_0 y}\}. \tag{6}$$

Here $C_\pm$ and $C_2$ are constants, $H$ is the Heaviside function, and the following estimates hold: $\psi_\pm^{(i)} = O(|z|^{-1})$, $|\nabla\psi_\pm^{(i)}| = O(|z|^{-2})$, $i = 1, 2$. When $S$ bounds $W^{(j)}$ internally the coefficients in (5) and (6) can be found as follows:

$$\mathcal{A} = 2C^{(j)}\int_S\left[u^{(j)}(x,y)\frac{\partial e^{\nu y}\cos\nu x}{\partial n} - e^{\nu y}\cos\nu x\frac{\partial u^{(j)}(x,y)}{\partial n}\right]ds,$$

$$\mathcal{A}_0 = 2\int_S\left[u^{(j)}(x,y)\frac{\partial C_0^{(j)}(y)\cos\nu_0 x}{\partial n} - C_0^{(j)}(y)\cos\nu_0 x\frac{\partial u^{(j)}(x,y)}{\partial n}\right]ds,$$

where $C^{(1)} = -(e^{2\nu h} + \varepsilon)^{-1}$, $C^{(2)} = -(1+\varepsilon)(e^{2\nu h} + \varepsilon)^{-1}$ and

$$C_0^{(1)}(y) = \frac{\nu\cosh\nu_0(y-h) + \nu_0\sinh\nu_0(y-h)}{(\nu-\nu_0)Q'(\nu_0)\cosh\nu_0 h}, \qquad C_0^{(2)}(y) = \frac{(1+\varepsilon)(\nu_0 - \nu\tanh(\nu_0 h))e^{\nu_0 y}}{(\nu - \nu_0)Q'(\nu_0)}.$$

By $Q'(\nu_0)$ we denote

$$\left.\frac{dQ}{dk}\right|_{k=\nu_0} = \frac{\varepsilon}{\varepsilon\nu - \nu_0}\left[\nu(1+\varepsilon) - \varepsilon h(\nu+\nu_0)^2 e^{-2\nu_0 h}\right].$$

The coefficients $\mathcal{B}$ and $\mathcal{B}_0$ can be obtained by virtue of replacing cos by $-$sin in the expressions for $\mathcal{A}$ and $\mathcal{A}_0$ respectively. Furthermore, if $S$ bounds internally $W^{(1)}$, then the relationship holds:

$$C_+ - C_- = \frac{-\varepsilon\nu}{1 + \varepsilon - \varepsilon\nu h}\int_S [u^{(1)}\frac{\partial x}{\partial n} - x\frac{\partial u^{(1)}}{\partial n}]\,ds.$$

Otherwise, $C_+ - C_- = 0$.

## 4. The wave-making resistance

Here we derive a wave resistance formula using the method proposed in [6]. It allows to express the resistance in terms of the coefficients in (5) and (6) not only when the body is totally immersed in one layer but also when it intersects the free surface or the interface, however those cases are out of this presentation. As the starting point we use the definition of resistance

$$R = -\int_S p\cos(n,x)\,ds, \tag{7}$$

where $p$ is the hydrodynamic pressure, which can be found from Bernoulli's integral $p = \text{const} - \rho_i g y - \rho_i V^{(i)2}/2$. Here the index $i$ is the number of the layer, in which the body moves, and $V^{(i)2}(x,y)$ is the square of absolute value of the fluid velocity, i.e. $V^{(i)2} = (u_x^{(i)} - U)^2 + (u_y^{(i)})^2$. Using (3), we get

$$R = \rho_i \int_S \left[2^{-1}|\nabla u^{(i)}|^2 \cos(n,x) - u_x^{(i)} \partial u^{(i)}/\partial n\right] ds. \tag{8}$$

Let us write the following Green formula

$$0 = \int_{W_\alpha^{(i)}} u_x^{(i)} \nabla^2 u^{(i)}\, dx\, dy = -\int_{W_\alpha^{(i)}} \nabla u^{(i)} \cdot \nabla u_x^{(i)}\, dx\, dy - \int_{\partial W_\alpha^{(i)}} u_x^{(i)} \frac{\partial u^{(i)}}{\partial n} ds,$$

where the normal $\vec{n}$ is directed into the domain $W_\alpha^{(i)} = R_\alpha^{(i)} \setminus \overline{B}$. Here $R_\alpha^{(1)} = \{|x| < \alpha,\ 0 < y < h\}$, $R_\alpha^{(2)} = \{|x| < \alpha,\ y < 0\}$ and $\alpha > \max\{|x| : (x,y) \in S\}$. By the divergence theorem we find from here that

$$-\int_S u_x^{(i)} \frac{\partial u^{(i)}}{\partial n} ds = \int_{\partial W_\alpha^{(i)} \setminus S} u_x^{(i)} \frac{\partial u^{(i)}}{\partial n} ds + 2^{-1} \int_{W_\alpha^{(i)}} (|\nabla u^{(i)}|^2)_x\, dx\, dy$$
$$= \int_{\partial W_\alpha^{(i)} \setminus S} u_x^{(i)} \frac{\partial u^{(i)}}{\partial n} ds - 2^{-1} \int_{\partial W_\alpha^{(i)}} |\nabla u^{(i)}|^2 \cos(n,x)\, ds.$$

Hence, we have

$$R = \rho_i \int_{\partial W_\alpha^{(i)} \setminus S} \left[u_x^{(i)} \frac{\partial u^{(i)}}{\partial n} - 2^{-1}|\nabla u^{(i)}|^2 \cos(n,x)\right] ds.$$

Also, one can obtain for $j \neq i$

$$0 = \rho_j \int_{\partial W_\alpha^{(j)}} \left[u_x^{(j)} \frac{\partial u^{(j)}}{\partial n} - 2^{-1}|\nabla u^{(j)}|^2 \cos(n,x)\right] ds.$$

Hence, the resistance can be rewritten as follows:

$$R = \sum_{i=1}^{2} \rho_i \int_{\partial W_\alpha^{(i)} \setminus S} \left[u_x^{(i)} \frac{\partial u^{(i)}}{\partial n} - 2^{-1}|\nabla u^{(i)}|^2 \cos(n,x)\right] ds.$$

The conditions on $F^{(1)}$ and $F^{(2)}$ in (1) and (2) allow to rewrite the integrals along the horizontal segments in the last formula in the form

$$(2\nu)^{-1}\rho_1 \left[(u_x^{(1)}(x,h))^2\right]_{x=-\alpha}^{x=\alpha}, \qquad \frac{1}{2\nu(\rho_2 - \rho_1)} \left[(\rho_1 u_x^{(1)}(x,0) - \rho_2 u_x^{(2)}(x,0))^2\right]_{x=-\alpha}^{x=\alpha}.$$

By virtue of the asymptotics (5) and (6) for these expressions and for the integrals along vertical segments after taking the limit as $\alpha \to +\infty$ we arrive at the following result:

$$R = -\frac{\nu\rho_1}{4}(\varepsilon + e^{2\nu h})\left(\mathcal{A}^2 + \mathcal{B}^2\right)$$
$$- H(\nu - \nu_*)\frac{\rho_2 \varepsilon(\nu - \nu_0)}{4}\left[\frac{\nu}{\nu + \nu_0} - \frac{\varepsilon h(\nu + \nu_0)e^{-2\nu_0 h}}{1 + \varepsilon}\right]\left(\mathcal{A}_0^2 + \mathcal{B}_0^2\right). \tag{9}$$

Here we have also applied the equality $e^{2\nu_0 h} = \varepsilon(\nu + \nu_0)/(\varepsilon(\nu - \nu_0) - 2\nu_0)$, which obviously follows from the definition of $\nu_0$ as the root of $Q(\nu_0) = 0$. It should be taken into account that the coefficients $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{A}_0$, $\mathcal{B}_0$ depend on the layer, in which the body moves.

Now we shall give another interpretation to the formula (9) in terms of energy transportation due to waves (cf [7], §26). Let

$$w_0^{(1)}(x,y) = \left[\left(1 + \varepsilon - \varepsilon\nu\nu_0^{-1}\right)\cosh(\nu_0 y) + \sinh(\nu_0 y)\right](\mathcal{A}_0 \sin\nu_0 x + \mathcal{B}_0 \cos\nu_0 x),$$
$$w_0^{(2)}(x,y) = e^{\nu_0 y}(\mathcal{A}_0 \sin\nu_0 x + \mathcal{B}_0 \cos\nu_0 x),\ w(x,y) = e^{\nu y}(\mathcal{A}\sin\nu x + \mathcal{B}\cos\nu x)$$

be the wave terms in the asymptotics (5) and (6) at infinity downstream. The mean energy per unit of th free surface which is transported away from the body by the waves of the wavenumber $\nu$ and $\nu_0$ is equal to

$$\bar{E} = \frac{\nu}{2\pi} \int_0^{2\pi/\nu} dx \left( \rho_1 \int_0^h |\nabla w|^2 \, dy + \rho_2 \int_{-\infty}^0 |\nabla w|^2 \, dy \right)$$

and

$$\bar{E}_0 = \frac{\nu_0}{2\pi} \int_0^{2\pi/\nu_0} dx \left( \rho_1 \int_0^h |\nabla w_0^{(1)}|^2 \, dy + \rho_2 \int_{-\infty}^0 |\nabla w_0^{(2)}|^2 \, dy \right)$$

respectively. Hence, we have

$$\bar{E} = \frac{\nu \rho_1}{2} (\varepsilon + e^{2\nu h}) \big( \mathcal{A}^2 + \mathcal{B}^2 \big), \qquad \bar{E}_0 = \frac{\nu \rho_2 \varepsilon (\nu - \nu_0)}{2(\nu + \nu_0)} \big( \mathcal{A}_0{}^2 + \mathcal{B}_0{}^2 \big).$$

For rewriting the formula (9) in terms of $\bar{E}$ and $\bar{E}_0$ we need the notion of group velocity which describes the speed of the energy transportation. If we denote by $\omega = \nu U$ and $\omega_0 = \nu_0 U$ the encounter frequencies of waves with different wavenumbers, then the corresponding group velocities are $C(\nu) = d\omega/d\nu$ and $C_0(\nu_0) = d\omega_0/d\nu_0$ respectively. Since $U = (g/\nu)^{1/2}$, then we have $C(\nu) = U/2$. When finding $C_0(\nu_0)$, we have to consider $\nu$ as the implicit function of $\nu_0$ defined by the equation $Q(\nu_0) = 0$. Finally, we can write

$$\frac{U - C_0(\nu_0)}{U} = \frac{1}{2} \left[ 1 - \frac{\varepsilon h (\nu + \nu_0)^2 e^{-2\nu_0 h}}{\nu (1 + \varepsilon)} \right].$$

Now, it is easy to verify that the formula (9) can be rewritten as follows:

$$R = -\frac{U - C(\nu)}{U} \bar{E} - H(\nu - \nu_*) \frac{U - C_0(\nu_0)}{U} \bar{E}_0,$$

what generalizes to the case of a two-layer fluid the formula which is given for the homogeneous fluid in [7].

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