(Abstract for the 11th Int. Workshop on Water Waves and Floating Bodies, March 1996)

Simulation of Shallow Water Waves Generated by Ships Using Boussinesq Equations Solved by a Flux-Difference-Splitting Method

Tao Jiang
Institute of Ship Technology
Mercator University, D-47048 Duisburg, Germany

Introduction

In recent papers by Chen and Sharma (1992) and Jiang, Sharma and Chen (1995) it has been shown that a modified Kadomtsev-Petviashili (KP) equation based on the shallow-water-wave theory can be successfully used for predicting the wave resistance and squat of a ship moving in a shallow-water channel. Although these investigations are valid in the subcritical, transcritical and supercritical speed range, they are restricted to a ship moving uniformly in a straight rectangular shallow-water channel.

The present study concentrates on the new development of an efficient algorithm using more general shallow-water-wave equations, namely, the well-known Boussinesq equations. Unlike the KP equation, the Boussinesq equations are also valid for unsteady cases caused, for example, by the hydrodynamic interactions between ships, by the geographical changes of water bottom, by the exciting waves near a coast, and so on. Due to the higher order term of the time derivative, however, their numerical approximation is not an easy task. Fortunately, by suitable transformation the Boussinesq equations acquire a form analogous to the well-solved Euler equations of 2-D gas dynamics, except for the higher order terms. Due to this mathematical analogy, a highly developed numerical method of aerodynamics can be implemented to solve the initial-boundary value problem governed by the Boussinesq equations. The main efforts are:

- Flux formulation of the Boussinesq equations
- Flux-difference-splitting based on the eigenvalue splitting
- High-order interpolation using MUSCL approach technique
- Conservative space discretization of the flux vectors
- Runge-Kutta time-stepping scheme

General Description

Consider a ship undergoing rectilinear motion with forward speed V in inviscid shallow water of depth h. Let $Oxyz$ be a shipbound Cartesian coordinate system centered at the midship with the plane Oxy lying on the undisturbed free surface, the $x-$axis pointing toward the bow, and the $z-$axis being positive upward. The irrotational flow generated by the moving ship can be thus described by a velocity potential Φ governed by the Laplace equation,

$$\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0, \tag{1}$$

in the whole fluid domain and by the following boundary conditions: first, kinematic and dynamic conditions,

$$\zeta_t - V\zeta_x + \Phi_x\zeta_x + \Phi_y\zeta_y = \Phi_z, \tag{2}$$

$$\Phi_t - V\Phi_x + \frac{|\nabla\Phi|^2}{2} + gz = 0, \tag{3}$$

respectively, on the free surface $z = \zeta(x, y, t)$, where g is the acceleration due to gravity; second, no-flux condition,

$$F_t - VF_x + \nabla\Phi \cdot \nabla F = 0, \tag{4}$$

on the hull-surface $F(x, y, z, t) = 0$; third, no-flux condition,

$$\Phi_z = 0, \tag{5}$$

on the water bottom $z = -h$; fourth, radiation condition,

$$\nabla\Phi \to 0, \tag{6}$$

at infinity $\sqrt{x^2 + y^2} \to \infty$.

Basic Solution Technique

If the flow problem considered here is further restricted by the relatively small water depth as well as by the slenderness of hull form, then the well-established technique of matched asymptotic expansions, first introduced by Tuck (1966), can be used. According to this approach the fluid-field is divided into an outer region (far field) and an inner region (near field) relative to the ship. In each flow region a scale analysis is then performed by selecting suitable scales for all variables, and the resulting simplified forms of the original governing equations are matched by means of asymptotic multiple-scale expansions in both regions.

The basic assumptions in the far field are that the waves generated by the ship are weakly nonlinear and long in comparison to water depth. These two features lead to a reasonable approximation of the 3-D potential Φ in terms of a depth-averaged 2-D potential φ in the horizontal plane as defined by

$$\varphi = \frac{1}{h + \zeta}\int_{-h}^{\zeta} \Phi(x, y, z, t)\mathrm{d}z, \tag{7}$$

where ζ denotes the local wave elevation.

Following standard shallow-water wave approximation (Taylor expansion in vertical direction) and using the 3-D Laplace equation as well as the no-flux condition at water bottom with constant water depth, the kinematic and dynamic conditions (2) and (3) on the free surface can be approximated by the well-known Boussinesq theory as follows:

$$\zeta_t - V\zeta_x + \nabla\cdot[(h + \zeta)\overline{\mathbf{u}}] = 0, \tag{8}$$

$$\overline{\mathbf{u}}_t - V\overline{\mathbf{u}}_x + \overline{\mathbf{u}}\cdot\nabla\overline{\mathbf{u}} + g\nabla\zeta = \frac{1}{3}h^2\nabla\nabla\cdot(\overline{\mathbf{u}}_t - V\overline{\mathbf{u}}_x), \tag{9}$$

where the ∇-operator from now on takes its 2-D form, e.g. $\nabla\varphi = (\varphi_x, \varphi_y)$, and $\overline{\mathbf{u}} = (\overline{u}, \overline{v}) = \nabla\varphi$. These so-called Boussinesq equations are finally closed by the boundary condition on the hull surface and radiation condition at infinity.

The boundary condition on the hull surface has to be determined by asymptotic outer expansions of the near-field 2-D potential in the vertical plane at each hull cross-section. After a lengthy derivation the no-flux boundary condition on the hull surface for the far-field flow is then reduced to

$$\overline{v}|_{y_o \to 0^{\pm}} = \mp\frac{1}{2}(h + \zeta_0)^{-1}[S(x, t)(V - \overline{u}_0)]_x. \tag{10}$$

In comparison with the classical slender-body theory this formulation takes also account of the effects of the longitudinal perturbation velocity $\overline{u}_0 = \frac{1}{2}[\overline{u}(x, 0^+, t) + \overline{u}(x, 0^-, t)]$ as well as of the instantaneous cross-section area $S(x, t)$ which depends on sinkage, trim and local wave elevation $\zeta_0 = \frac{1}{2}[\zeta(x, 0^+, t) + \zeta(x, 0^-, t)]$.

Flux Formulation

Substituting the linearized expression $\overline{\mathbf{u}}_t - V\overline{\mathbf{u}}_x = -g\nabla\zeta$ of the equation (9) into its high-order term on the right hand side, a numerically advantageous formulation of the Boussinesq equations is obtained:

$$H_t + [uH]_x + [vH]_y = 0, \tag{11}$$

$$u_t + uu_x + vu_y + gH_x = -\frac{1}{3}gh^2(\nabla^2 H)_x, \tag{12}$$

$$v_t + uv_x + vv_y + gH_y = -\frac{1}{3}gh^2(\nabla^2 H)_y, \tag{13}$$

where $H = h+\zeta$ denotes the local water depth. $u = \overline{u}-V$ and $v = \overline{v}$ are the longitudinal and transversal components of the relative velocity between the ship and ambient water. From the mathematical point of view, this formulation is analogous to that governing 2-D gas dynamics, and, hence, suitable for applying highly developed numerical methods of aerodynamics. For this purpose, the Boussinesq equations (11, 12, 13) are finally transformed to the following flux-vector formulation:

$$\mathbf{Q}_t + \mathbf{F}_x + \mathbf{G}_y = \mathbf{E}, \tag{14}$$

where $\mathbf{Q}$ is the conservative variable vector, $\mathbf{F}$ and $\mathbf{G}$ are the flux vectors, and $\mathbf{E}$ denotes the proper high-order term. These vectors are defined as follows:

$$\mathbf{Q} = \begin{pmatrix} H \\ uH \\ vH \end{pmatrix}, \qquad \mathbf{F} = \begin{pmatrix} uH \\ uHu + \frac{1}{2}gH^2 \\ uHv \end{pmatrix},$$

$$\mathbf{G} = \begin{pmatrix} vH \\ vHu \\ vHv + \frac{1}{2}gH^2 \end{pmatrix}, \qquad \mathbf{E} = \begin{pmatrix} 0 \\ -\frac{1}{3}gh^2H(\nabla^2 H)_x \\ -\frac{1}{3}gh^2H(\nabla^2 H)_y \end{pmatrix}.$$

The associated Jacobian matrices of the flux-vectors $\mathbf{F}$ and $\mathbf{G}$ are given by

$$\mathbf{A} = \frac{\partial \mathbf{F}}{\partial \mathbf{Q}} = \begin{bmatrix} 0 & 1 & 0 \\ -u^2 + gH & 2u & 0 \\ -uv & v & u \end{bmatrix}, \qquad \mathbf{B} = \frac{\partial \mathbf{G}}{\partial \mathbf{Q}} = \begin{bmatrix} 0 & 0 & 1 \\ -uv & v & u \\ -v^2 + gH & 0 & 2v \end{bmatrix}. \tag{15}$$

The corresponding eigenvalue matrices and eigenvector matrices are determined as follows:

$$\Lambda_{\mathbf{A}} = \begin{bmatrix} u & 0 & 0 \\ 0 & u - \sqrt{gH} & 0 \\ 0 & 0 & u + \sqrt{gH} \end{bmatrix}, \quad \mathbf{T_A} = \begin{bmatrix} 0 & v^{-1} & v^{-1} \\ 0 & (u - \sqrt{gH})v^{-1} & (u + \sqrt{gH})v^{-1} \\ 1 & 1 & 1 \end{bmatrix}, \tag{16}$$

$$\Lambda_{\mathbf{B}} = \begin{bmatrix} v & 0 & 0 \\ 0 & v - \sqrt{gH} & 0 \\ 0 & 0 & v + \sqrt{gH} \end{bmatrix}, \quad \mathbf{T_B} = \begin{bmatrix} 0 & (v - \sqrt{gH})^{-1} & (v + \sqrt{gH})^{-1} \\ 1 & u(v - \sqrt{gH})^{-1} & u(v + \sqrt{gH})^{-1} \\ 0 & 1 & 1 \end{bmatrix}. \tag{17}$$

Based on these real eigenvalues of the matrices $\mathbf{A}$ and $\mathbf{B}$, the flow involved can be thus classified as a subcritical flow for $u^2 + v^2 < gH$ and as a supercritical flow for $u^2 + v^2 > gH$.

Difference Schemes

To solve the Boussinesq equations efficiently, the explicit M-step Runge-Kutta method, which was successfully applied to Euler equations by Jameson et al. (1981) as well as to Navier-Stokes equations by Hänel and Breuer(1990), is implemented hier for time discretization:

$$\left.\begin{aligned} \mathbf{Q}^{(0)} &= \mathbf{Q}^n, \\ \mathbf{Q}^{(m)} &= \mathbf{Q}^0 - \alpha_m \,\Delta t\ \mathrm{Res}(\mathbf{Q}^{(m-1)}) \\ \mathbf{Q}^{(n+1)} &= \mathbf{Q}^M. \end{aligned}\right\} \qquad m = 1, 2, .., M, \tag{18}$$

For $M = 5$ the matched coefficients take the values of $\alpha_m = (0.059, 0.145, 0.273, 0.5, 1.0)$ that lead to the maximum Courant number for an upwind scheme. $\mathrm{Res}(\mathbf{Q}^n)$ denotes the residue resulting from a conservative space discretization at nth time step. For instance, at the i,j-node it reads:

$$\mathrm{Res}_{i,j}(\mathbf{Q}^n) = \frac{\tilde{\mathbf{F}}_{i+\frac{1}{2},j} - \tilde{\mathbf{F}}_{i-\frac{1}{2},j}}{x_{i+\frac{1}{2},j} - x_{i-\frac{1}{2},j}} + \frac{\tilde{\mathbf{G}}_{i,j+\frac{1}{2}} - \tilde{\mathbf{G}}_{i,j-\frac{1}{2}}}{y_{i,j+\frac{1}{2}} - y_{i,j-\frac{1}{2}}} - \mathbf{E}_{i,j}. \tag{19}$$

According to the flux-difference-splitting (FDS) scheme by Roe (1981), the numerical flux vector $\tilde{\mathbf{F}}$ is approximated by

$$\tilde{\mathbf{F}}_{i+\frac{1}{2},j} = \frac{1}{2}[\mathbf{F}(\mathbf{Q}^+_{i+\frac{1}{2},j}) + \mathbf{F}(\mathbf{Q}^-_{i+\frac{1}{2},j})] + \frac{1}{2}\overline{\mathbf{A}}(\overline{\mathbf{Q}}_{i+\frac{1}{2},j})(\mathbf{Q}^+_{i+\frac{1}{2},j} - \mathbf{Q}^-_{i+\frac{1}{2},j}), \tag{20}$$

with the associated matrix $\overline{\mathbf{A}}$ having only positive eigenvalues:

$$\overline{\mathbf{A}} = \mathbf{T_A}|\Lambda_\mathbf{A}|\mathbf{T_A}^{-1}. \tag{21}$$

The corresponding numerical flux vector $\tilde{\mathbf{G}}$ can be written in the same manner.

Using the MUSCL approach technique, the numerical flux at the cell interface is defined by the variable vector $\mathbf{Q}$ either interpolated or extrapolated to the cell interface. The general formulation is given by

$$\begin{aligned}
\mathbf{Q}^+_{i+\frac{1}{2},j} &= \mathbf{Q}^n_{i,j} + \frac{1}{4}\varepsilon_i[(1+\kappa)(\mathbf{Q}^n_{i+1,j} - \mathbf{Q}^n_{i,j}) + (1-\kappa)(\mathbf{Q}^n_{i,j} - \mathbf{Q}^n_{i-1,j})\frac{x_{i+1,j} - x_{i,j}}{x_{i,j} - x_{i-1,j}}], \\
\mathbf{Q}^-_{i+\frac{1}{2},j} &= \mathbf{Q}^n_{i+1,j} - \frac{1}{4}\varepsilon_{i+1}[(1+\kappa)(\mathbf{Q}^n_{i+1,j} - \mathbf{Q}^n_{i,j}) + (1-\kappa)(\mathbf{Q}^n_{i+2,j} - \mathbf{Q}^n_{i+1,j})\frac{x_{i+1,j} - x_{i,j}}{x_{i+2,j} - x_{i+1,j}}], \\
\overline{\mathbf{Q}}_{i+\frac{1}{2},j} &= \frac{1}{2}(\mathbf{Q}^+_{i+\frac{1}{2},j} + \mathbf{Q}^-_{i+\frac{1}{2},j}),
\end{aligned} \tag{22}$$

with the local limiter function ε_i to avoid numerical oscillation and the upwind control parameter κ of value $\frac{1}{3}$ leading to a third-order space discretization. The term $\mathbf{E}_{i,j}$ involves first a central difference scheme for $\nabla^2 H$ and then an upwind difference scheme for $\nabla(\nabla^2 H)$ which can be treated simularly as for the flux vectors.

Numerical Example

To examine the numerical method introduced here for solving the Boussinesq equations, it is intended in the primary stage to investigate the wavemaking problem of a Series 60 hull moving uniformly in a straight rectangular shallow water-channel. The no-flux condition on the channel sidewalls have to be additionally satisfied. At the upstream and downstream open boundaries a suitable Sommerfeld-like radiation condition is applied. Numerical results from the present theory as well as their comparisons with those from the previous calculations and model experiments will be presented at the workshop.

References

Chen, X.-N. and Sharma, S.D. 1992, "A Slender Ship Moving at a Near-critical Speed in a Shallow Channel", Presented at the 18th International Congress of Theoretical and Applied Mechanics, Haifa, Israel, Published in Journal of Fluid Mechanics (1995), Vol. 291, pp. 263-285.

Hänel, D. and Breuer, M. 1990, " Numerical Solution of the Incompressible Navier-Stokes Equations for Unsteady Threedimensional Flow", Proceedings of the Third International Congress on Fluid Mechanics, Cairo, Egypt.

Jameson, A., Schmidt, W. and Turkel, E. 1981, "Numerical Solutions of the Euler Equations by Finite Volume Methods Using Runge-Kutta Time-Stepping Schemes", AIAA paper 81-1259, AIAA 14th Fluid and Plasma Dynamics Conference, California, USA.

Jiang, T., Sharma, S.D. and Chen, X.-N. 1995, "On the Wavemaking, Resistance and Squat of a Catamaran Moving at High Speed in a Shallow Water Channel", Proceedings of the 3rd International Conference on Fast Sea Transportation FAST'95, Lübeck-Travemünde, Germany, pp. 1313-1325.

Roe, P.L. 1981, "Approximate Riemann Solvers, Parameter Vectors, and Difference Schemes", Journal Computational Physics, Vol. 43, pp. 357-372.

Tuck, E.O. 1966, "Shallow-Water Flows past Slender Bodies", Journal of Fluid Mechanics, Vol. 26, pp. 81-95.