Second-order interaction between waves and multiple bottom-mounted vertical circular cylinders

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This paper presents a semi-analytical method for studying the second-order diffraction of bichromatic waves by multiple bottom-fixed circular cylinders. Unlike in some earlier analyses of second order interactions, we do not use a plane wave approximation, adopting instead the exact interaction theory of Linton & Evans (1990) to obtain the first-order and linear components of second-order potentials. A particular solution which exactly satisfies the inhomogeneous free-surface boundary condition is sought, based on a Green-function approach. The free-surface integrals are performed in terms of local polar coordinate systems which coincide with the individual cylinders, and are transformed into one-dimension by using the Fourier expansion method. Some results for the force quadratic transfer functions are given, based on direct integration of the second order potential. The formulation can also yield the influence of interaction effects on the free surface elevation.

1. Decomposition of the velocity potential

We consider the diffraction of a plane bichromatic incident wave by a group of vertical circular cylinders fixed on the sea bottom. Under the assumption of irrotational flow, a velocity potential can be defined to describe the wave field in the whole fluid domain. By adopting the conventional Stokes perturbation procedure, this potential can be expressed as:

$$\begin{aligned}\Phi(r,\theta,z,t) &= \epsilon\Phi^{(1)}+\epsilon^2\Phi^{(2)}+O(\epsilon^3)=\Re\{\sum_{j=1}^{2}\phi_j^{(1)}(r,\theta,z)e^{-i\omega_j t}+\\ &\sum_{j=1}^{2}\sum_{l=1}^{2}[\,\phi_{jl}^{-}(r,\theta,z)e^{-i\omega^- t}+\phi_{jl}^{+}(r,\theta,z)e^{-i\omega^+ t}\}+O(\epsilon^3)\end{aligned}\qquad(1)$$

where $\omega^{\pm}=\omega_j\pm\omega_l$. Each of the first-order and the second-order sum and difference-frequency potential components should satisfy the governing Laplace equation in the fluid domain, and the zero normal velocity condition on the seabed and on each of the cylinder surfaces. The first-order potential satisfies a homogeneous free-surface condition while the sum and difference-frequency potentials satisfy inhomogeneous free-surface conditions at the corresponding frequencies.

We decompose the first and the second-order potentials into the following components:

$$\phi_j^{(1)}=\phi_{Ij}^{(1)}+\phi_{Dj}^{(1)},\quad \phi_{jl}^{\pm}=\phi_{I,jl}^{\pm}+\phi_{D,jl}^{\pm},\qquad(2)$$

where subscripts I,D denote incident and diffraction potentials respectively. We further decompose the second-order diffraction potential into three components: a linear component which corresponds to the second-order

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incident wave (at the sum and difference frequencies), a nonlinear (locked wave) component induced by the first-order forcing on the free surface, and a second linear component (free-wave component) associated with the nonlinear part. This third component is required to enforce the impermeable body-surface condition. Therefore we can write (for simplicity we omit the subscripts j, l hereafter):

$$\phi_D^{\pm}(x,y,z) = \phi_{D1}^{\pm}(x,y,z) + \phi_{D2}^{\pm}(x,y,z) + \phi_{D3}^{\pm}(x,y,z). \tag{3}$$

2 Interaction theory for linear components

The s-th cylinder has radius a_s and its axis has horizontal coordinates (x_{cs}, y_{cs}) in the global system. In terms of the s-th local cylindrical polar system, (r_s, θ_s), we can write the j-th ($j = 1,2$) first-order incident and diffraction potentials as follows:

$$\phi_{Ij}^{(1)} = I_{sj} \sum_{n=-\infty}^{\infty} J_n(k_{wj} r_s) e^{in(\frac{\pi}{2}-\beta)} e^{in\theta_s} Z(z), \quad I_{sj} = e^{ik_{wj}(x_{cs}\cos\beta + y_{cs}\sin\beta)}; \tag{4}$$

$$\phi_{Dj}^{(1)} = \sum_{n=-\infty}^{\infty} A_{nj}^{s} \frac{H_n(k_{wj} r_s)}{H_n'(k_{wj} a_s)} e^{in\theta_s} Z(z); \tag{5}$$

where k_{wj} is the j-th first-order wave number, and β is the angle of incidence. The form of $\phi_{Dj}^{(1)}$ is slightly different from that of Linton & Evans (1990), and yields a linear system of equations with a diagonally dominant coefficient matrix (which is desirable for numerical implementation). By applying the body boundary condition on the q-th cylinder surface, and using the Bessel addition theorem, one can obtain:

$$A_{mj}^{q} + \sum_{s=1, s\neq q}^{N_c} \sum_{n=-\infty}^{\infty} A_{nj}^{s} \frac{H_{n-m}(k_{wj} R_{sq})}{H_n'(k_{wj} a_q)} J_m'(k_{wj} a_q) e^{i(n-m)\alpha_{sq}} \delta_{sq} = -I_q e^{im(\frac{\pi}{2}-\beta)} J_m'(k_{wj} a_q), \tag{6}$$

where

$$\begin{aligned} \alpha_{sq} &= \tan^{-1}[(y_{cq}-y_{cs})/(x_{cq}-x_{cs})], & \delta_{sq} &= 1, & x_{cq} &> x_{cs}; \\ \alpha_{sq} &= -\tan^{-1}[(y_{cq}-y_{cs})/(x_{cq}-x_{cs})], & \delta_{sq} &= (-1)^{m-n}, & x_{cq} &< x_{cs}; \end{aligned}$$

and N_c is the number of cylinders. The above expression is very convenient for numerical implementation. We remark that the same interaction approach is also applicable to the second-order linear components, though details will not be presented here.

3 Nonlinear component of the second-order potential

We can obtain a particular solution to the second-order diffracted potential , $\phi_{D2}^{\pm}$, which satisfies the inhomogeneous free surface boundary condition, and a Neumann condition on the surface of the qth cylinder. Based on Green's second identity, $\phi_{D2}^{\pm}$ can be expressed as:

$$\phi_{D2}^{\pm}(r_q, \theta_q, z) = \frac{1}{2\pi} \int\int_{S_F} Q^{\pm}(s) G^{\pm}(r_q, \theta_q, z; s) ds \tag{7}$$

where S_F is the entire quiescent free-surface excluding the water planes of all the cylinders. To facilitate a more efficient numerical algorithm, we expand both $Q^{\pm}(\xi_q, \eta_q)$ and $G^{\pm}(r_q, \theta_q, z; \xi_q, \eta_q)$ into Fourier components, and apply an eigen-function expansion in z to obtain:

$$\phi_{D2}^{\pm}(r_q, \theta_q, z) = \sum_{n=-\infty}^{\infty} e^{in\theta_q} \{ \sum_{m=0}^{\infty} Z_m^{\pm}(0) Z_m^{\pm}(z) \int_{a_q}^{\infty} \xi Q_n^{\pm}(\xi) G_{mn}^{\pm}(r_q, \xi) d\xi \}. \tag{8}$$

4 Second-order 'free-wave' component $\phi_{D3}^{\pm}$

An appropriate form for the second-order 'free-wave' component, which satisfies the homogeneous boundary condition on the free surface, would be:

$$\phi_{D3}^{\pm} = \sum_{l=1}^{N_c} \sum_{n=-\infty}^{\infty} \sum_{q=0}^{\infty} C_{nq}^{l} U_n(k_q^{\pm} r_l) Z_q(z) e^{in\theta_l}. \tag{9}$$

By applying the boundary condition on the j-th cylinder surface and using the same procedure as used in section 2, one can derive:

$$C_{m0}^{j} + \sum_{l=1,l\neq j}^{N_c} \sum_{n=-\infty}^{\infty} C_{n0}^{l} \frac{H_{n-m}(k_0^{\pm} R_{lj})}{H_n'(k_0^{\pm} a_j)} J_m'(k_0^{\pm} a_j) e^{i(n-m)\alpha_{lj}} \delta_{lj} = P_{j0} \tag{10}$$

$$C_{mq}^{j} + \sum_{l=1,l\neq j}^{N_c} \sum_{n=-\infty}^{\infty} C_{nq}^{l} \frac{K_{n-m}(k_q^{\pm} R_{lj})}{K_n'(k_q^{\pm} a_j)} I_m'(k_q^{\pm} a_j) e^{i(n-m)\alpha_{lj}} \delta_{lj} = P_{jq} \tag{11}$$

where $q = 1, 2, 3, \cdots$, P_{j0}, P_{jq} are known functions. Superscript j denotes the j-th cylinder on which the global coordinate system is fixed, and R_{lj} is the distance between the corresponding cylinders.

5 Free-surface integrals

The intervals of the free-surface integrals, in terms of the qth local polar coordinate system, are divided into three parts: the near-field region $r_q < R_{min}$ (where $R_{min} = min\{R_{sq}\}$, $j,k = 1,2,\cdots N_c$), the intermediate part and the far-field part. In the near-field region, the free-surface forcing function $Q_n^{\pm}$ possesses a simple form, due to the simple expression for $\phi_j^{(1)}$ resulting from the Bessel addition theorem (for details see Linton & Evans, 1990). This form, however, is not valid in the intermediate and far-field regions. For the intermediate region, we use a Fourier expansion in the circumferential direction and a quadratic interpolation in the radial direction to separate the variables (r,θ) in the forcing functions. In the far-field region, the local evanescent terms in the velocity potentials are neglected, and the free-surface integrals can be performed analytically (Kim & Yue 1989, Chau & Eatock Taylor 1992).

6 Results

Some contour plots of the quadratic transfer functions in the bi-frequency plane are given in the figures, for one and four cylinders. The centres of the cylinders of radius a are at the corners of a square of side $5a$, and the water depth is $4a$. The results have been calculated at intervals $\Delta\nu_i = 0.2$, $(i = 1,2)$, where $\nu_i = {\omega_i}^2/g$. Intermediate results were obtained by use of bi-cubic splines. More results will be presented in the workshop and compared with those obtained by other solutions.

References

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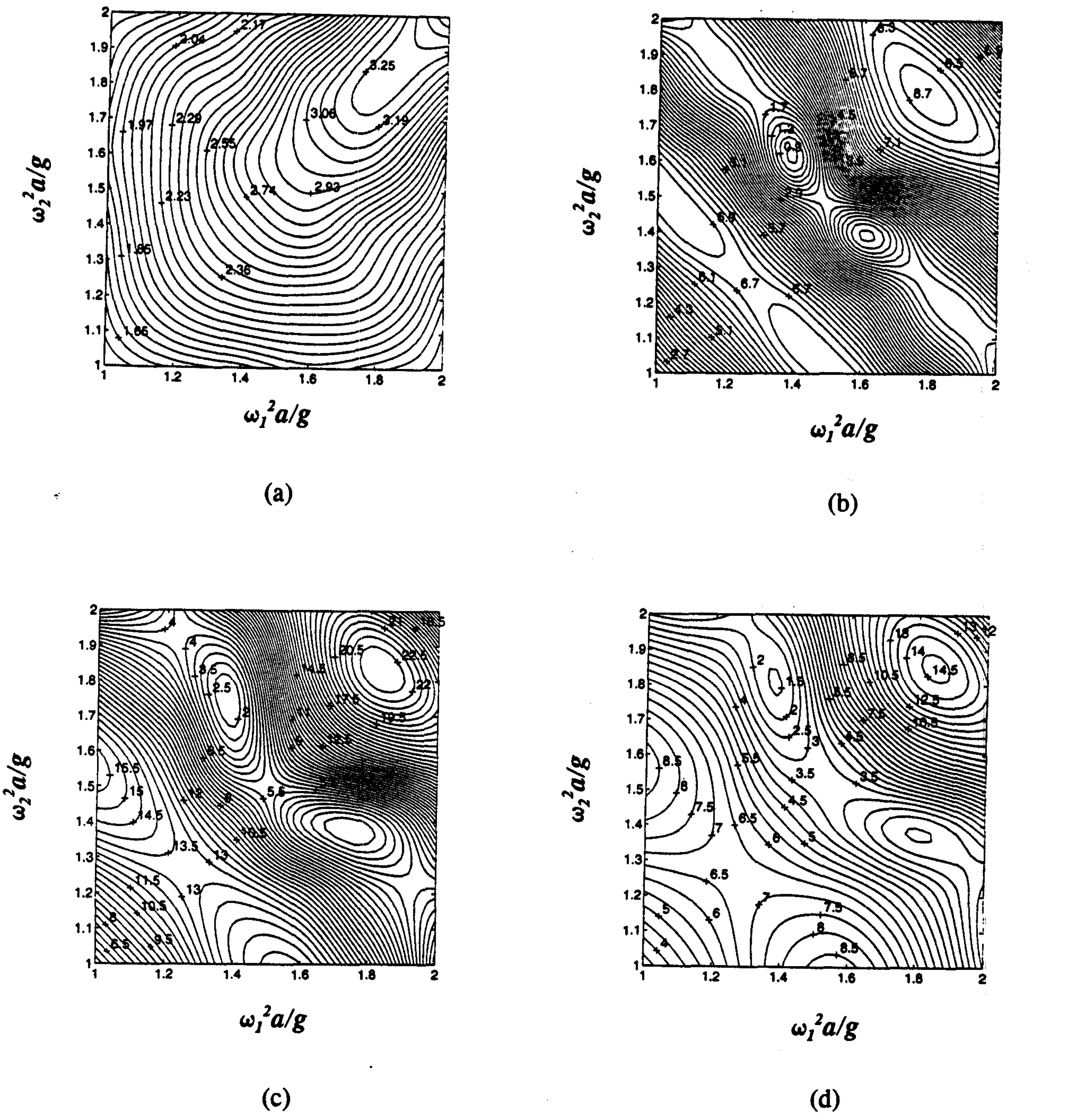

Contours of magnitudes of second-order sum-frequency surge force QTF components: (a) total sum-frequency QTF on a single cylinder; (b) quadratic component of QTF on 4 cylinders; (b) sum-frequency potential contribution to QTF on 4 cylinders; (c) total sum-frequency QTF on 4 cylinders.