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A DIFFERENTIAL METHOD FOR THE MODELLING OF WAVE TRANSMISSION, REFLECTION AND ABSORPTION BY "NUMERICAL BEACHES"

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INTRODUCTION

The free-surface absorbing layer or "numerical beach" is one of the most common method for absorbing outgoing waves at the end of numerical wave basins. It is based on a paper by Israeli & Orszag [1], and was applied to free-surface flow simulations for the first time by Baker et al. [2]. Later on, many other authors further developed this technique which consists in adding an additional dissipative term either in the dynamic free-surface condition [3], or in the kinematic condition [4], or in both of them [2] [5] [6]. Furthermore, this technique can be coupled to a Neumann boundary condition on the vertical closing surface [7] [8] to extend the bandwidth of the overall absorption in the low frequency range without increasing the length of the numerical beach. The form of the added term remains arbitrary, provided it results in energy dissipation when the waves pass through the "beach" zone ② (Fig1.). It seems however that no systematic study was carried out to optimize the technique, taking advantage of this degree of freedom. The purpose of the present preliminary study was to develop a method for computing the entire solution (velocity potential, reflection transmission and absorption coefficients) for a given form of the additional term in the linearized frequency domain. This method is based on a differential formulation of the problem with regard to the horizontal variable X and it could be used in other applications where the wavenumber depends on X such as : wave reflection from thin floating ice sheet [9], or variable finite depth problems without any wide spacing assumption as in [10] [11]. It will be used in a future study to determine the optimal form of the dissipative term, again in the frequency domain, with the constraint of keeping the beach as short as possible for obious numerical reasons. Finally, we shall implement this optimal solution in our non-linear time domain "numerical wave tank" [12] [13] where we will quantify its absorption efficiency by performing numerical experiments.

MATHEMATICAL FORMULATION AND SOLUTION METHOD:

The water depth is assumed to be the same in both the left and right semi-infinite band ① and ③ ; the space variables (X,Y) are nondimensionalized with respect to depth. A two dimensional, irrotational potential linear theory is assumed.

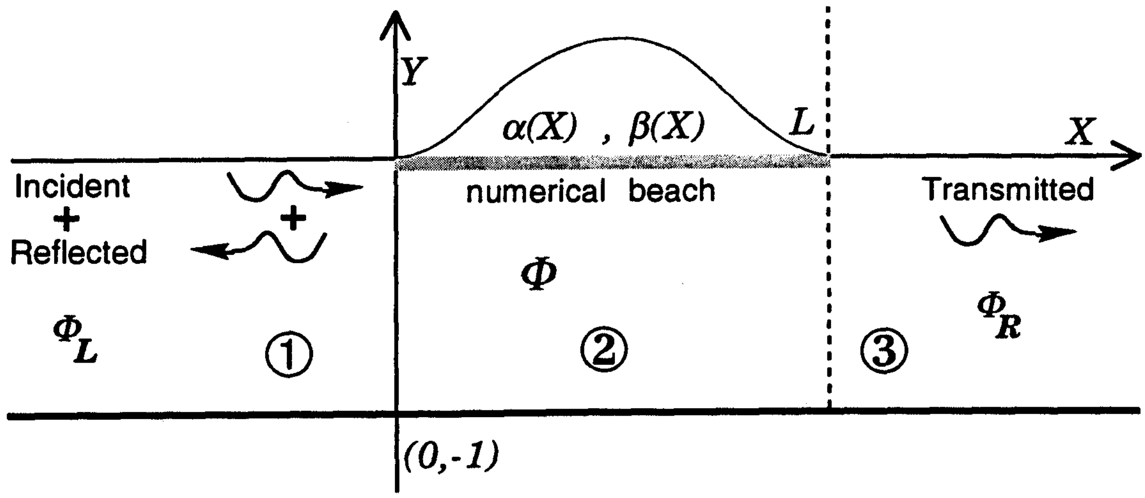

Fig.1: Definition sketch

A rightgoing incident wave is given by an Airy potential with amplitude L_I (in (1)). The usual linearized free-surface boundary conditions hold in ① and ③ where the real coefficients m_n satisfy the standard dispersion relationship : $m_0 \tanh(m_0) = \omega^2$ and $m_k \tan(m_k) = -\omega^2$.

Left domain ① :

$$\Phi_L = \left(L_I e^{-im_0 X} + L_0 e^{im_0 X}\right)\cosh(m_0(\omega)(Y+1)) + \sum_{n=1}^{\infty} L_n e^{im_n X}\cos(m_n(\omega)(Y+1)) \quad (1)$$

Right domain ③ :

$$\Phi_R = R_0 e^{-im_0 X}\cosh(m_0(\omega)(Y+1)) + \sum_{n=1}^{\infty} R_n e^{-im_n X}\cos(m_n(\omega)(Y+1)) \quad (2)$$

The transmission and reflection coefficients of the beach are defined resp. as :

$$t = \left|\frac{R_0}{L_I}\right| \qquad r = \left|\frac{L_0}{L_I}\right| \quad (3)$$

Beach zone ② :

In this transition domain $0 < X < L$ the free-surface conditions is modified by addition of dissipative terms:

$$\begin{cases} \dfrac{\partial \Phi}{\partial t} = -Y - \alpha(X)\Phi - \gamma(X)\dfrac{\partial \Phi}{\partial Y} \\ \dfrac{\partial Y}{\partial t} = \dfrac{\partial \Phi}{\partial Y} - \beta(X)Y \end{cases} \quad (4)$$

where $\alpha(X)$, $\beta(X)$ and $\gamma(X)$ are given functions (to be optimized later) defining the damping beach. Both cases were tested, in which the dissipative term in the dynamic condition was assumed proportional either to the potential or to the normal (vertical) velocity, alone, or together with the modified kinematic condition.

The dispersion relationship now becomes : $M_k(\omega,X)\tan(M_k(\omega,X)) = Q(\omega,X)$ (5)

with : $$Q(\omega,X) = \frac{(-\omega^2 + \alpha(X)\beta(X)) + i\omega(\alpha(X)+\beta(X))}{1+\beta(X)\gamma(X)+i\omega\gamma(X)}$$

Solutions M_k of Eq. (5) are now complex and depend on the space variable X ; this prevents us from finding the potential in this zone using the method of separation of variables. However, provided all the M_k's satisfy (5), the potential $\Phi(X,Y)$ in the damping domain ② can still be sought in the form :

$$\Phi(X,Y) = \sum_{k=0}^{\infty} \Phi_k(X)\cos\left[M_k(X).(Y+1)\right] \quad (6)$$

Defined that way, the potential satisfies the free surface and the sea bottom boundary conditions. The Laplace's equation now leads to an infinite set of 2nd order ordinary differential equation in the X variable which can be written in a matrix form :

$$\frac{d^2[\Phi]}{dX^2} = [\mathbf{B}]\frac{d[\Phi]}{dX} + [\mathbf{C}][\Phi] \quad (7)$$

with, in vector notation : $^T[\Phi] = [\Phi_0, \Phi_1,.., \Phi_k]$

Given a frequency ω, the matrices **B** and **C** depend on X only through the functions $\alpha(X)$, $\beta(X)$ and $\gamma(X)$, and may be computed from them at any location $0<X<L$. The solution is then obtained by solving (7) as an initial value problem by any appropriate ODE integration method (a fourth order Runge-Kutta method was used in the present study). Initial conditions are given by the potential (1) and its horizontal gradient at the input abscissa $X=0$; the latter is defined by the (k+2) coefficients $[L_I, L_0,..L_k]$, where k is the truncation order of the infinite system (7). The problem being entirely linear, it is solved (k+2) times, with vector basic solutions $[\varphi(X)]_j$ corresponding to the j^{th} problem : $[0,..,L_j = 1,..,0]$. The unknown potential in the beach domain ② may finally be computed as a linear combination of the $[\varphi(X)]_j$ with coefficients $[L_I, L_0,..L_k]$, which are themselves determined by enforcing the continuity of the potential and its horizontal derivative at the output section : $X=L$. From the velocity potential in ②, the mean energy flux across the numerical beach is computed directly as :

$$E_a = \frac{1}{T}\int_0^T dt \int_0^L \left[\frac{\partial \Phi}{\partial T}\left(\frac{\partial Y}{\partial T} - \frac{\partial \Phi}{\partial Y}\right) + P\frac{\partial Y}{\partial T}\right]_{Y=0} dX \qquad (8)$$

Let the mean energy flux of the incident wave be denoted by E_I, then the beach absorption coefficient a may be defined by the ratio : $a = (E_a/E_I)^{1/2}$, and the energy balance for the domain ② requires :

$$a^2 = 1 - r^2 - t^2 . \qquad (9)$$

SOME RESULTS

The above method was applied to beaches of fixed length L=2, with various form for functions $\alpha(X)$, $\beta(X)$ and $\gamma(X)$ (all symmetric around X=1) : step, linear, quadratic and cubic functions. We have also addressed the case of shorter beaches of length 1 combined with a vertical reflecting wall at X=1 (Fig.4 and 5).

For all cases, we tested both $\alpha\Phi$ and $\gamma\Phi_Y$ pressure terms in (4), alone or associated with a βY mass flux term. Due to the limited space, only a few results are reported in the present abstract, but more results will be presented at the conference. We chose to give herein the absorption, reflection and transmission coefficients for beaches with no modification of the kinematic condition (i.e : $\beta(X) = 0$; $\forall X$) in two configurations : beach alone with a semi-infinite band on each side (Fig2, 3), and the case of a beach adjacent to a vertical wall at $X = L$ (Fig.4, 5)

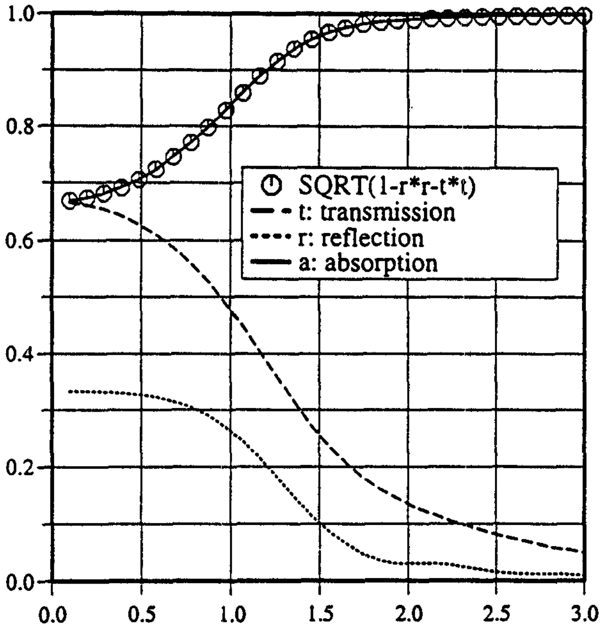

Fig2 : coefficients for a "potential beach" (*i.e* extra term=$\alpha\Phi$, $\beta(X) = 0$, $\gamma(X) = 0$): L=2 ; $\alpha(X)$ a symmetrical linear function ("chinese hat")) : $\alpha(0) = 0.$ $\alpha(L/2) = 1.$ $\alpha(L) = 0.$

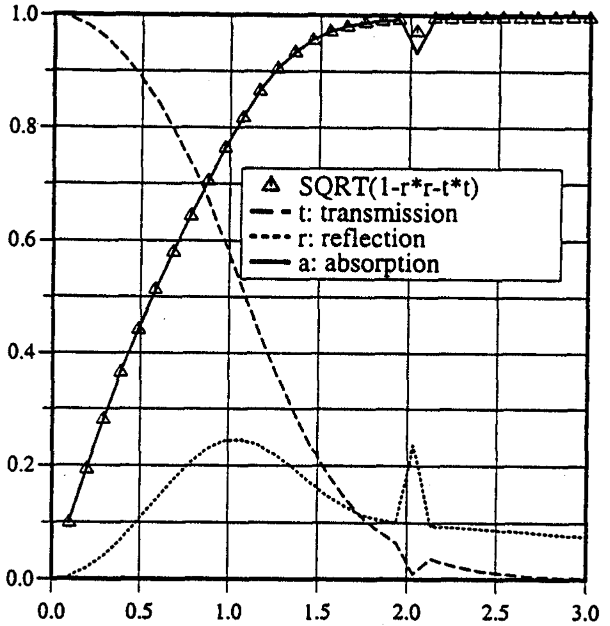

Fig3 : coefficients for a "velocity beach" (*i.e* extra term=$\gamma\Phi_Y$, $\beta(X) = 0$, $\alpha(X) = 0$): L=2 ; $\gamma(X)$ a symmetrical linear function : $\gamma(0) = 0.$ $\gamma(L/2) = 1.$ $\gamma(L) = 0.$

Comparing Fig.2 and 3, we see that the main difference between the two kinds of absorption via the dynamic free-surface condition lies in the low frequency domain where absorption and reflection tend to zero with the frequency using a velocity term (Fig.3), while both of these tend toward a common non-zero value using a potential term (Fig.2). Hence, the latter choice seems to be the best, at least when the beach is used alone. When the beach is associated with a piston-like Neuman condition on X=L as in [7] [8], the absorption of low frequency waves ($\approx \omega$<1.0) is handled by the piston in such a way that we have to optimize the numerical beach in the high frequency range only. From this viewpoint, we see that when the frequency increases the reflection coefficient falls to zero much more rapidly with a velocity term than with a potential term. Thus the choice may depend on the absorption strategy and is left open to the user. The accuracy obtained by the present differential method is highlighted by the coincidence of the dots and the solid line which are respectively the rigth and left hand sides of the energy balance relationship Eq. (9). The agreement is always better than four digits with a truncation order as low as k=5, except for a few localized frequencies where the solution seems singular.

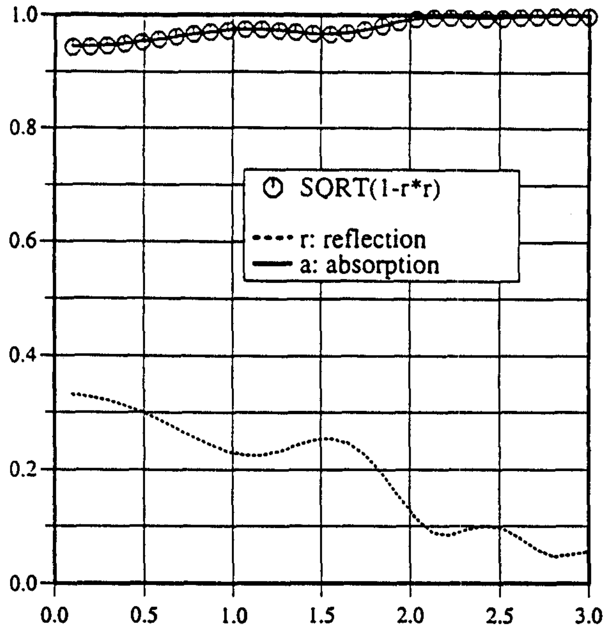

Fig4 : coefficients for a "potential beach" (*i.e* extra term= $\alpha\Phi$, $\beta(X)=0$, $\gamma(X)=0$): associated with a vertical wall at $X=L=1$; $\alpha(X)$ a linear function : $\alpha(0)=0.$ $\alpha(L)=1.$

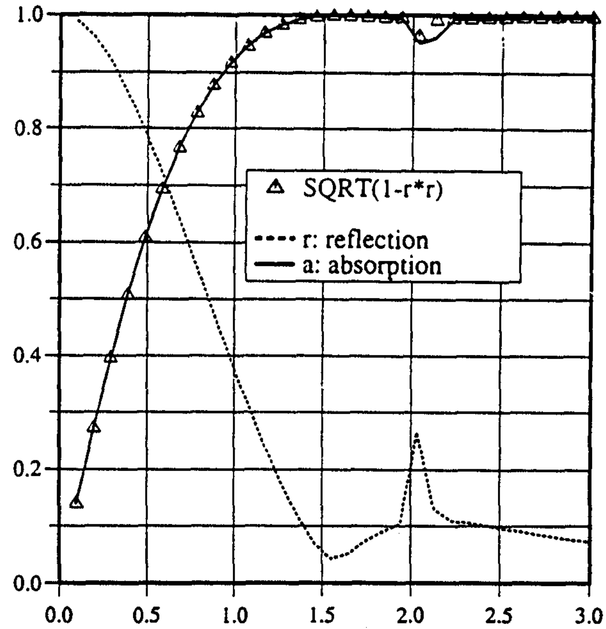

Fig5 : coefficients for a "velocity beach" (*i.e* extra term= $\gamma\Phi_Y$, $\beta(X)=0$, $\alpha(X)=0$): associated with a vertical wall at $X=L=1$; $\gamma(X)$ a linear function : $\gamma(0)=0.$ $\gamma(L)=1.$

At the moment, we suspect this to be a purely numerical divergence, but a more serious mathematical problem is not excluded, and we are working on fixing the problem. The same phenomena appears on Fig.5 where the beach is adjacent to a vertical wall. When this point is cleared up, we shall optimize the numerical beach using the present method, with the constraint of keeping its length as small as possible in order to save *cpu* time and memory size in numerical wave tanks.

REFERENCES

[1]M ISRAELI S.A. ORSZAG, Approximation of radiation boundary conditions, *J. Comp. Physics*, 41, pp115-135 (1981).

[2] G.R. BAKER, D.I. MEIRON, S.A. ORSZAG: Application of a Generalized Vortex Method to non-linear Free Surface flows. 3rd Int. Conf. Numer. Ship Hydrod. Paris 1981.

[3] Y. CAO, R.F BECK, W. SCHULTZ: *An absorbing Beach for Numerical Simulations of Nonlinear Waves in a Wave Tank* 1993 -8th WWWFB. St John's; Newfoundland.

[4] D.E NAKOS, Rankine Panel Methods for Transient Free Surface Flows.

[5] R. COINTE, Nonlinear Simulation of Transient Free Surface Flows, *Proc. 5th Int. Conf. Num. Ship Hydrodynamics, Hiroshima, Japan* Sept. 1989.

[6] J.G. TELSTE, Inviscid Flow about a cylinder rising to a free surface, *J. Fluid. Mec*, **182**, pp149-168. (1987)

[7] A. CLEMENT, J.F DOMGIN: *Wave absorption in a 2D numerical wave basin by coupling two methods,* April 1995 - Proc. 10th Int. WWWFB; Oxford, pp 15/20.

[8] A. CLEMENT: *Coupling of Two Absorbing Boundary Conditions for 2D Time-domain Simulations of Free Surface Gravity Waves.* (to appear in J. Comp. Physics 1996) .

[9] M. WEITZ, J.B. KELLER (1950): *Reflection of Water Waves from Floating Ice in Water of Finite depth.* Comm. Pure Appl. Maths, **3**, pp. 305-318.

[10] P. DEVILLARD, F. DUNLOP, B. SOUILLARD (1988): *Localization of gravity waves on a channel with a random bottom.* J. Fluid Mech., 186, pp. 521-538.

[11] D. EVANS, C.M. LINTON (1994): *On step approximations for water-wave problems.* J. Fluid Mech., 278, pp. 229-249.

[12] A. CLEMENT *The Diffraction of a solitary wave by a free surface piercing cylinder.* 6th WWWFB. Woods Hole - Massachusetts. 1991

[13] A. CLEMENT, S. MAS: *Hydrodynamic Forces Induced by a Solitary Wave on a Submerged Cylinder.* 1995 - Proc. ISOPE95, The Hague, Vol.III pp339/347.