Infinite systems of equations. A note on the effect of truncation.

by F.Ursell
Department of Mathematics, University of Manchester,
Manchester M13 9PL, U.K.

The solution of many important problems of linear ship hydrodynamics involves the solution of infinite systems of linear equations. A typical example is the heaving motion of a half-immersed circular cylinder of radius a, involving a positive frequency parameter $\theta = \omega^2 a/g$, where ω is the angular frequency. It is known that for each value of θ this can be reduced to the solution of an infinite system of the form

$$x_m(\theta) + \theta \sum_{n=1}^{\infty} k_{mn} x_n(\theta) = a_m(\theta), \; m = 1, \cdots, \infty,$$

where the coefficients

$$k_{mn} = \frac{32}{\pi} \frac{m^2(n-1)}{(4m^2-1)(2n-1)(2n+2m-1)(2m-2n+1)},$$

are independent of θ, and where $a_m(\theta)$ is a double integral satisfying an inequality of the form $|a_m(\theta)| < M(\theta)/m^2$. For each value of θ we wish to find the sums

$$S_1(\theta) = \sum_{m=1}^{\infty} \frac{x_m(\theta)}{m(2m-1)} \text{ and } S_2(\theta) = \sum_{m=1}^{\infty} \frac{x_m(\theta)}{m(4m^2-1)},$$

which occur in the expressions for the virtual mass and damping. In practice the infinite system is truncated, i.e. it is replaced by a finite $N \times N$ system, and the corresponding infinite sums are replaced by finite sums of N terms. We are thus led to consider the following problem: We consider an infinite system of equations of the form

$$x_m + \sum_{n=1}^{\infty} k_{mn} x_n = a_m, 1 \le m < \infty,$$

or in matrix notation

$$(I + K)x = a.$$

where I denotes the infinite unit matrix, and where the elements k_{mn} of the matrix K and the components a_n of the column vector a are given. With the solution $\{x_n\}$ of this system we associate the sum

$$S = \sum_{n=1}^{\infty} b_n x_n = b'x.$$

In practice this system is replaced by the finite *truncated system* involving N unknowns,

$$\xi_m(N) + \sum_{n=1}^{N} k_{mn}\xi_n(N) = a_m, 1 \leq m \leq N,$$

or in matrix notation

$$[(I+K)]_N[\xi(N)]_N = [a]_N,$$

with the associated sum

$$S(N) = \sum_{n=1}^{N} b_n\xi_n(N) = [b]'_N[\xi(N)]_N,$$

where the symbol $[a]_N$ denotes a column vector of dimension N and the symbol $[A]_N$ denotes a matrix of dimension $N \times N$. We wish to find how $S(N)$ approaches the limit S, i.e. we wish to find the *truncation error*

$$\Delta(N) = S - S(N) = b'(I+K)^{-1}a - [b]'_N([I+K]_N)^{-1}[a]_N$$

as a function of the large parameter N. The elements of the inverse matrices $(I+K)^{-1}$ and $([I+K]_N)^{-1}$ cannot be found explicitly even when the elements of $(I+K)$ are known explicitly. It will be seen that an explicit bound can always be found for $\Delta(N)$ but this bound may not give the true order of magnitude. We would like to find the true order of magnitude of $\Delta(N)$, or even the asymptotic dependence of $\Delta(N)$ on N .

I have found that some progress can be made. For some infinite systems, e.g. the heaving circular cylinder, it is possible to find the form of the asymptotic behaviour of x_M when $M \to \infty$, and this can then be used to improve the approximation for $\Delta(N)$.

The theory is illustrated by means of the example

$$x_m - \sum_{n=1}^{\infty} \frac{x_n}{mn(m+n+1)} = \frac{1}{m^2}, \ 1 \leq m < \infty.$$

with the associated sum

$$S = \sum x_n/n^2.$$

It is shown that

$$\Delta(N) \sim \frac{\alpha}{N^3} + \frac{\beta}{N^4} + O\left(\frac{1}{N^{9/2}}\right),$$

where α and β are certain constants ; the bound mentioned earlier gives

$$\Delta(N) = O\left(\frac{1}{N^{3/2}}\right).$$

The methods used in the present paper are elementary, involving at most Schwarz's Inequality $(\sum X_m Y_m)^2 \leq (\sum X_m^2)(\sum Y_m^2)$.

DISCUSSION

Yeung, R. W. : I would like to suggest that perhaps a simpler problem, but one that may still yield insights into the convergence characteristics of such truncated systems, is to consider the right-hand column to behave like m^{-p} and the kernel function K_{mn} to be separable of the form $m^{-q}n^{-r}$. If $\Delta(N)$ can be worked out or bounded for b_n of the form n^{-s}, then one can learn how the system under truncation behaves with (p, q, r, s) as general parameters. I think solution of such a simplified problem will still be very useful to the field.

Ursell, F.: For coefficients in the form of a finite series

$$k_{mn} = \sum_{j=1}^{l} f_j(m) g_j(n)$$

the infinite system and the truncated system can both be solved explicitly. For clearly each unknown x_m in the infinite system must be of the form

$$x_m = \sum_{1}^{l} A_j f_j(m) + a_m,$$

where the constants A_j are determined by an $l x l$ system. The truncated system can be treated similarly. Approximations for $\Delta(N)$ can therefore be obtained. However, I have not been able to adopt this idea to coefficients like $k_{mn} = 1/(m+n+1)$.

Rainey, R. T. C.: I wonder if I might take advantage of Fritz Ursell's presence here in Oxford to show a picture of Oxford University's most famous son, Dr. Samuel Johnson, (1709-1784), and of a definition from Johnson's famous *Dictionary* (1755).

I believe we have a similar problem of definition at the moment in the water waves community, over the use of the word *order*, and the notation $A = O(B)$. See table below.

	STRICT SENSE	COLLOQUIAL SENSE	POSSIBLE CONVENTION
A is of order B.	**In some limiting process,** A is ultimately proportional to B (i.e. $A/B \rightarrow$ constant)	$A \approx B$	COLLOQUIAL sense, unless stated otherwise.
$A = O(B)$	**In some limiting process,** A is ultimately no more than proportional to B (i.e. $A/B < \infty$)	$A \approx B$	STRICT sense, unless stated otherwise.

IMMATE'RIAL. *adj.* [*immateriel*, Fr. *in* and *materia*, Lat.]

1. Incoporeal; distinct from matter; void of matter.

Angels are spirits *immaterial* and intellectual, the glorious inhabitants of those sacred palaces, where there is nothing but light and immortality; no shadow of matter for tears, discontentments, griefs, and uncomfortable passions to work upon; but all joy, tranquility and peace, even for ever and ever, do dwell. *Hooker.*

As then the soul a substance hath alone,
Besides the body, in which she is confined;
So hath she not a body of her own,
But is a spirit, and *immaterial* mind. *Davies.*

Those *immaterial* felicities we expect, suggest the necessity of preparing our appetites, without which heaven can be no heaven to us. *Decay of Piety.*

No man that owns the existence of an infinite spirit can doubt the possibility of a finite spirit; that is, such a thing as is *immaterial*, and does not contain any principle of corruption. *Tillotson.*

2. Unimportant; without weight; impertinent; without relation. This sense has crept into the conversation and writings of barbarians; but ought to be utterly rejected.

The colloquial sense of the word *order* is so widespread, particularly in engineering, that I believe Prof. Ursell's exemplary adherence to the strict sense of the word is now unusual in our community, and we must expect the colloquial sense to be assumed, unless we state otherwise.

On the other hand I feel the same way about the colloquial usage of the notation $A = O(B)$, as Johnson did about the modern (and now standard!) usage of the word *immaterial.* Does Prof. Ursell agree with me that we ought to be able to assume that $A = O(B)$ is meant in the strict sense, unless we state otherwise?

Ursell, F.: Rod Rainey claims that the verbal expression "A is of order B" differs in meaning from the symbolic expression " $A = O(B)$". This claim is not generally accepted by mathematicians. I have consulted a very authoritative text [1, ch.1 §2] from which I take the following, slightly abbreviated:

If $|f(x)/\phi(x)|$ is bounded, we write

$$f(x) = O\{\phi(x)\}, (x \to \infty), \qquad \text{(A)}$$

or $f = O(\phi)$; in words, *f is of order not exceeding ϕ* .

Next, the symbol O is sometimes associated with an interval $[a, \infty)$ instead of the limit point ∞. Thus

$$f(x) = O\{\phi(x)\}, \text{ when } x \in [a\ \infty)$$

simply means that $|f(x)/\phi(x)|$ is bounded throughout $a \le x < \infty$. The notation $O(\phi)$ can also be used to denote the classes of functions with the property (A) or *unspecified* functions with this property.

This quotation from Olver shows that "when A is of order B" does not imply (to mathematicians) that the ratio A/B tends to a limit l; this would be denoted by $A \sim lB$. There is a symbolic notation, "$A \asymp B$" , which indicates that A and B are of exactly the same order, i.e. that $c_1 B < A < c_2 B$, where c_1 and c_2 are non-zero constants, but this notation is used only in Number Theory, see [2, ch.1, §1.6], and is not generally familiar even to mathematicians.

References:

[1] F.W.J. Olver, Asymptotics and Special Functions, Academic Press, 1974.

[2] G. H. Hardy and E. M. Wright, An introduction to the theory of numbers, Oxford University Press, 1938.

[*Note by Editor: This exchange prompted an extensive correspondence between Rainey and Ursell, not on the strict definition of "A = O(B)" on which both agreed, but on the meaning of "A is of order B"*].