A Simple Idea to Calculate Shallow Water Flow with Steep Waves

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Introduction

Shallow water equations are often used to calculate the water flow in tanks and on the decks of rolling ships. If the waves are smooth this problem is easy to solve with finite difference or finite volume schemes. But the computation should also be stable if the surface of the fluid becomes steep. In order to ensure this, several authors propose the Glimm method, flux splitting methods or other methods with strong numerical damping. Here a simple, stable algorithm with low damping is described.

Discrete Equations

We start with equations for the conservation of mass and momentum for a finite volume Ω with surface S:

$$\underbrace{\frac{\partial}{\partial t}\int_{\Omega} \rho d\Omega}_{A} + \underbrace{\int_{S} \rho(\vec{v}\vec{n}) dS}_{B} = 0 \quad , \qquad \underbrace{\frac{\partial}{\partial t}\int_{\Omega} \rho\vec{v} d\Omega}_{C} + \underbrace{\int_{S} (\rho\vec{v})(\vec{v}\vec{n}) dS}_{D} = \vec{f} \quad . \tag{1}$$

ρ is the fluid density, $\vec{n}$ the unit outward normal on S, $\vec{f}$ is the force acting on Ω and $\vec{v}$ is the fluid velocity.

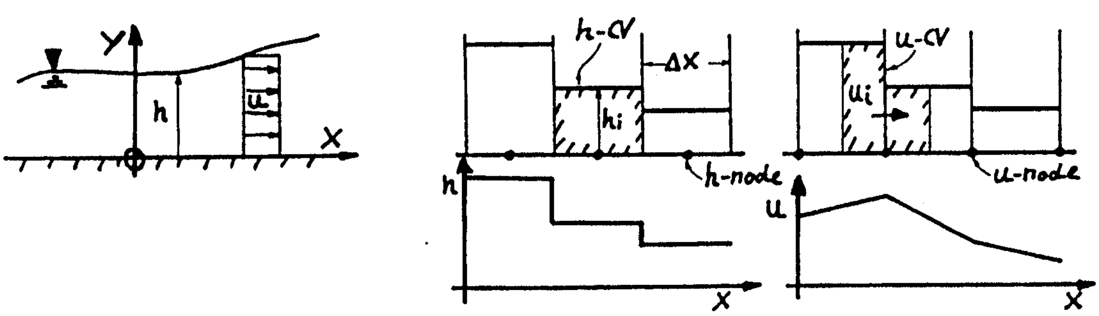

Figure 1: Definition and discretisation

We consider a one-dimensional flow with surface elevation h and velocity u (the average value from the bottom to the surface), which are functions of space and time coordinates x and t. ρ is constant, and the force is caused only by hydrostatic pressure (accleration of gravity g).

To find a discrete form of the equations over x we use a staggered net as shown in Figure 1. The height h in the control volume of mass (h-CV) is assumed constant. For the velocity u, a linear function between the u-nodes is assumed. In the control volume of momentum (u-CV) the velocity is piecewise linear. The velocity at the faces of the u-CV and the height h at the faces of the h-CV are approximated with the upwind-scheme. Δx and Δt are the finite steps in space resp. in time. The index i follows the direction of x.

The time integration is approximated with the trapezoidal scheme (also known as Crank-Nicolson scheme). We denote by upper index 0 and 1 the time t and $t+\Delta t$ respectively. We obtain the following discrete form of the equations (1):

$$\begin{aligned}(A^1 - A^0)/\Delta t \;&+\; \frac{1}{2}(B^0 + B^1) = 0 \quad , \\ (C^1 - C^0)/\Delta t \;&+\; \frac{1}{2}(D^0 + D^1 - f^0 - f^1) = 0 \quad . \end{aligned} \tag{2}$$

The terms in equations (2) are:

$$
\begin{aligned}
A &= h_i \Delta x \quad , \\
B &= -h_{i-1}\max(u_i,0) + h_i\max(-u_i,0) + h_i\max(u_{i+1},0) - h_{i+1}\max(-u_{i+1},0) \quad , \\
C &= \left(u_i(\frac{3}{8}h_i + \frac{3}{8}h_{i-1}) + u_{i-1}\frac{1}{8}h_{i-1} + u_{i+1}\frac{1}{8}h_i \right) \Delta x \quad , \\
D &= -u_{i-1}\max(-m_w,0) + u_i\max(m_w,0) + u_i\max(m_e,0) - u_{i+1}\max(-m_e,0) \quad , \\
f &= \vec{f} = \frac{1}{2}g({h_{i-1}}^2 - {h_i}^2) \quad , \\
& m_w = -\frac{1}{2}(u_{i-1} + u_i)h_{i-1} \quad , \qquad m_e = \frac{1}{2}(u_i + u_{i+1})h_i \quad .
\end{aligned}
$$

We solve equations (2) for the unknown u^1 and h^1 with the Gauss-Seidel scheme. It was found in many test cases that 5-7 iteration cycles give good results. A slight underrelaxation (0.8 ...0.9) acclerates the convergence.

Examples

Dam-breaking: Initially the fluid is at rest and the height is $h = 0.05m$ at the left side of the dam, where it is much greater than at the right side ($h = 0.01m$). At $t = 0$ the dam is removed. Figure 2 shows the surface at $t = 1.2s$. The dashed line is the analytic solution of the one-dimensional shallow-water equations for this case (see [Stoker]).

Moving tank: We consider a tank (length $1.2m$, water depth initially constant $h_0 = 0.09m$) rolling harmonically about the z-axis (see figure 3). The coordinate system is fixed in the bottom of the tank; the roll amplitude a is small ($\varphi = a\sin(\omega t)$). The linearised force in x-direction on the u-CV i is: $-\varphi\, g\, \Delta x\, 0.5(h_{i-1} + h_i)$. We chose $\omega = 2.46/s$ and $a = 0.0349$rad, and after some periods a bore is observed. Figure 3 compares the computed height h at the middle of the tank with that measured by [Verhagen and Wijngaarden].

If we chose $a = 0.3$rad the bottom of the tank gets dry. No numerical instability occurs (Figure 3). The amplitude is no longer small; for more acurate results, the forces to the u-CV should be chosen as [Dillingham] or [Huang and Hsiung] did.

Conclusion

The described scheme is simple, and gives acceptable results of the shallow water equations also if a hydraulic jump occurs. In the test cases a steep (and not smeared) shock front is computed.

References

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[Huang and Hsiung] Z. J. Huang, C. C. Hsiung: Application of the Flux Difference Splitting Method to Compute Nonlinear Shallow Water Flow on Deck; 9th Int. Workshop on Water Waves and Floating Bodies, 1994 pp. 63-66.

[Stoker] J. J. Stoker: Water Waves; Interscience Publishers, Inc., New York, 1957, pp. 333-341.

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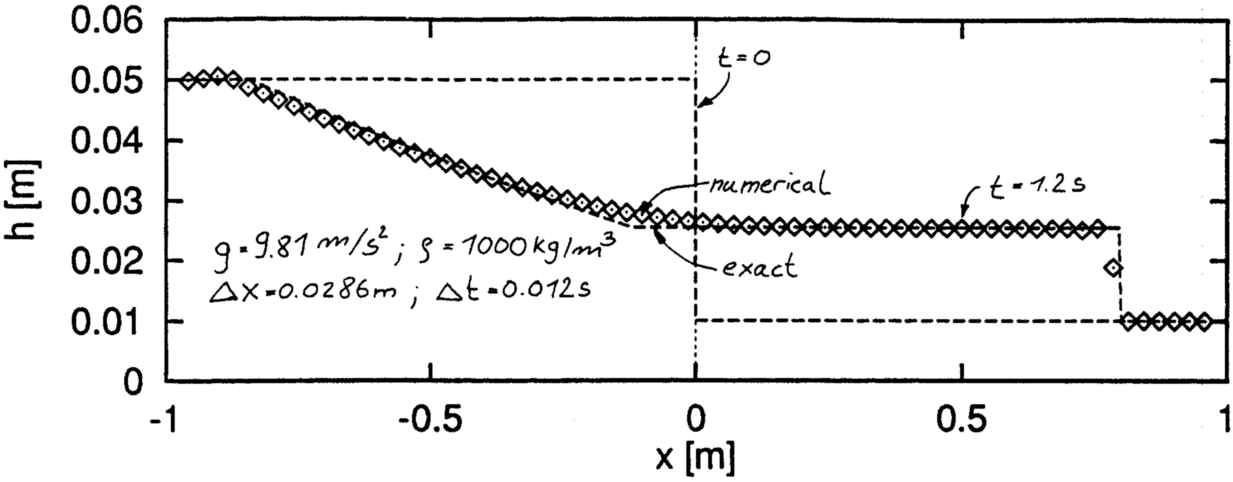

Figure 2: Breaking of a dam

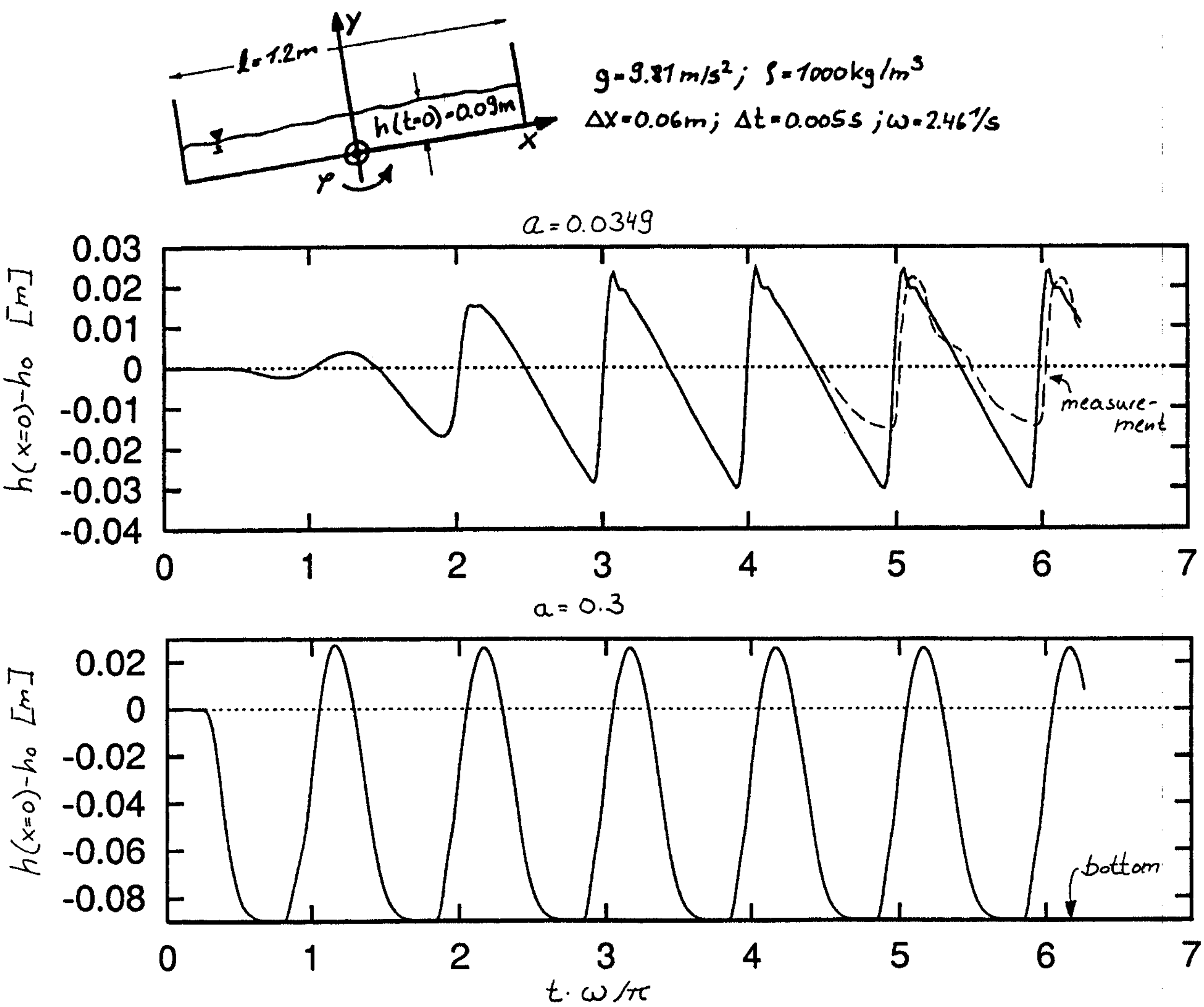

Figure 3: h in the middle of the oscillating tank

DISCUSSION

Hsiung, C. C.: 1) Did you repeat the experiments by Adee and Caglayan (1982) to examine your test results?

2) How do you treat the problem when the bottom is partially dry?

Schumann, C.: 1) No, the plotted experimental data are from Verhagen and Wiyngaarden, please see the last reference.

2) It works automatically in this scheme, just a fraction of the fluid volume remains in the "dry" CV (approx. 10^{-6} of the total fluid volume in the example of figure 3).

Martin, P. A. : Is shallow-water theory appropriate for all the applications that you have in mind?

A comment: I want to draw attention to some work done on the water-on-deck problem at Manchester in the early 1980s, by A. F. Jones and A. Hulme; they have a paper in J. Ship Research.

Schumann, C.: I have in mind to use it in cases when the wave length, λ, is large compared to the mean water depth, h, for example $h/\lambda < 0.1$. But in the problems where you find a bore I don't know how to define the wave length. So the border where the theory is useful, and where not, cannot be drawn by a simple formula. Only experience can make you sure that the results are valid. I close with a question: if you never had heard anything about potential theory, would you have thought that you could calculate the drag and lift of an airfoil quite well?

Kim, Y. W..: In the front of a hydraulic jump, the speed of fluid motion is very fast. We may experience difficulties because of stability. Have you ever checked the stability condition for this kind of jump? How much will be the Courant number ($C\Delta t/\Delta x$) in this case?

Schumann, C.: I made some tests and it was found that $CFL = \max\,(|u| + \sqrt{gh})/(\Delta x/\Delta t))$ should be lower than ≈ 0.5. I did no linear stability analysis.