COMPUTATION OF THE FINITE DEPTH TIME-DOMAIN GREEN FUNCTION IN THE LARGE TIME RANGE.

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The first formulation of the time-domain Green function seems to appear in Brard in 1948 in the case of infinite water depth. Finkelstein, in 1957, gave the expressions for finite and infinite water depth in two and three dimensions. Since that time, a lot of formulations have been developed in order to evaluate this function in the infinite water depth case (Jami 1982, Newman 1985, Liapis 1986, King 1987). Other methods were proposed to avoid in-line calculations of this function: the tabulation of the function (Ferrant 1988) and the identification of the function (Clément 1991-1992). Owing of these studies, the time-domain modelization of the seekeeping problem has become feasible, at least when the water depth may be considered to be infinite.

On the contrary, in the case of finite water depth, only few formulations of the time-domain Green function have been developed up to now. Indeed, this function is far more difficult to evaluate numerically and fewer problems require its use. So, we began the study of this problem three years ago and the first results we have obtained were presented last year during the ninth *Workshop on Water Waves and Floating Bodies* in Kuju. In this paper, the Green function was expressed as odd powers series of the non dimensional time variable T. This formulation is only valid for small T due to the convergence of the coefficients of the series.
In the present paper, alternative formulations of the finite water depth time-domain Green function are proposed for large values of the time variable. They are based on the method proposed in Clarisse, Newman, Ursell (1994), which will be referred as CNU method in the following. In these formulations the parameter R/T plays a central role, where R is the non dimensional horizontal distance between the source point and the field point. Two formulations have been developed depending on the value of the parameter with regard to the maximum group velocity $V_g = 1$ (in the present linear theory). When $R/T < 1$, the initial real CNU method has been used. When $R/T > 1$, a complex CNU method has been developed.

THE MEMORY PART OF THE GREEN FUNCTION.

We are concerned here with the potential generated in $M(X,Y,Z)$ at time T by a source of unit impulsive strength $\delta(0)$ located at $M'(X',Y',Z')$. This Green function is the sum of a impulsive term and a memory part given by:

$$G(M,M',T) = \int_0^{+\infty} f(K)\frac{\sin(Tw)}{w} KJ_0(KR)dK \quad (1) \text{ with} \begin{cases} w = \sqrt{K\tanh K} \\ f(K) = \dfrac{\cosh K(Y+1)\cosh K(Y'+1)}{\cosh^2 K} \end{cases} (2)$$

, $R = \sqrt{(X-X')^2 + (Z-Z')^2}$ being the horizontal distance between the two points. Distances are nondimensionalized with respect to the constant depth h.

The range of integration is extended by introducing the Hankel function of second order $H_0^{(2)}$, and we use the following integral representation:

$$H_0^{(2)}(KR) = \frac{\sqrt{2}i}{\pi}\int_{-\infty}^{+\infty} \frac{\exp\left[-iKR\left(1+\sigma^2\right)\right]}{\sqrt{1+\sigma^2/2}} d\sigma \quad (3)$$

Hence: $$G(M,M',T)=\frac{2^{-3/2}}{\pi}\int_{-\infty}^{+\infty} dK \int_{-\infty}^{+\infty} \frac{K}{w} f(K) \frac{\exp\left[iT\left(w-aK-aK\sigma^2\right)\right]}{\sqrt{1+\frac{\sigma^2}{2}}} d\sigma \quad (4)$$

where $a=R/T$.

REAL CNU METHOD: $R/T<1$.

Then we make a double change of variables of integration, in accordance with the CNU method:
$\begin{cases}\sqrt{K\tanh K}-aK=\varepsilon u-\frac{u^3}{6}\\ K\sigma^2=uv^2\end{cases}$ (5). By differentiating the first equation with respect to K, we obtain ε in terms of the stationary point K_0 such as $w'(K_0)=a$: $\varepsilon=\frac{3^{2/3}}{2}\left(\sqrt{K_0\tanh K_0}-aK_0\right)^{2/3}$. Equation $w'(K_0)=a$ has real solutions when $a<1$. Then this method is only valid in that case. A second change of variables leads to a polynomial phase function:
$\begin{cases}u=\alpha(\xi+\eta)\\ v=\beta(\xi-\eta)\end{cases}$ (6) with $\begin{cases}\alpha=-2^{-1/3}\\ \beta=-2^{-5/6}a^{-1/2}\end{cases}$. Therefore, the Green function can be expressed as

$$\begin{cases}G(M,M',T)=\frac{2^{-5/3}}{\pi\sqrt{a}}\int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} d\eta G^*(\xi,n)\exp(iT\psi)\\ G^*(\xi,\eta)=f(K)\frac{\varepsilon-u^2/2}{w'-a}\sqrt{\frac{Ku}{\left(K+uv^2/2\right)\tanh K}} \quad (7)\\ \psi(\xi,\eta;\varepsilon)=\frac{1}{3}\xi^3+\frac{1}{3}\eta^3+\alpha\varepsilon\xi+\alpha\varepsilon\eta\end{cases}$$

In terms of the new set of variables (ξ,η), four saddle points symmetrically located at $\xi=\pm\xi_0$ and $\eta=\pm\xi_0=\pm\sqrt{-\alpha\varepsilon}$ can be evaluated. Now, by using two Bleistein sequences such as:

$$\begin{cases}G^*=E_0+A_0\xi+B_0\eta+C_0\xi\eta+\left(\xi^2-\xi_0^2\right)H_0(\xi,\eta)+\left(\eta^2-\xi_0^2\right)K_0(\xi,\eta)\\ \frac{\partial H_0}{\partial\xi}+\frac{\partial K_0}{\partial\eta}=E_1+A_1\xi+B_1\eta+C_1\xi\eta+\left(\xi^2-\xi_0^2\right)H_1(\xi,\eta)+\left(\eta^2-\xi_0^2\right)K_1(\xi,\eta)\end{cases} \quad (8),$$

and after some computations, we show that the first order and the second order of the double integral in (7) can be evaluated owing to values of G^* and its partial derivatives with respect to u and to v at the saddle points.

Finally, we obtain: $G(M,M',T)=\frac{2^{1/3}\pi}{T^{2/3}\sqrt{a}}\left[E_0A_i^2-T^{-2/3}A_i'^2C_0+2T^{-4/3}A_1A_iA_i'\right]$ (9), where the first and the second terms are the leading terms, and the third one, a first order correction with respect to T^{-1}. A_i is the Airy function and A_i' its derivative. In (9), their argument is $-\xi_0^2T^{2/3}$.

COMPLEX CNU METHOD: $R/T>1$.

From the equation (4), we pass in the complex domain by using two successive changes of variables and we obtain:

$$G(M,M',T)=-\frac{2^{-3/2}}{\pi}\int_{-i\infty}^{+i\infty} d\sigma \int_{-i\infty}^{+i\infty} dK \frac{Kf(iK)}{\sqrt{K\tan K}} \frac{\exp\left[T\left(-\sqrt{K\tan K}+Ka+Ka\sigma^2\right)\right]}{\sqrt{1-\sigma^2/2}} \quad (10)$$

Then using the same methodology as previously expressed, we make the changes of variables as follows:

$$\begin{cases} -\sqrt{K\tan K} + aK = \varepsilon u - \dfrac{u^3}{6} \\ K\sigma^2 = uv^2 \end{cases} \text{(11)et} \begin{cases} u = \alpha(\xi+\eta) \\ v = \beta(\xi-\eta) \end{cases} \text{(12)}$$

Equation $\tilde{w}(K) = a$ has real solutions when $a > 1$. Then this method is only valid in that case.

We now obtain the following system:

$$\begin{cases} G(M,M',T) = -\dfrac{2^{-5/3}}{\pi\sqrt{a}} \displaystyle\int_{-i\infty}^{+i\infty} d\xi \int_{-i\infty}^{+i\infty} d\eta G^*(\xi,\eta)\exp(T\psi) \\ G^*(\xi,\eta) = \dfrac{Kf(iK)}{\tilde{w}} \dfrac{\varepsilon - u^2/2}{a-\tilde{w}'} \sqrt{\dfrac{u}{K}} \dfrac{1}{\sqrt{1-\sigma^2/2}} \\ \psi = -\varepsilon 2^{-1/3}(\xi+\eta) + \dfrac{1}{3}\left(\xi^3+\eta^3\right) \end{cases} \qquad (13)$$

with $\tilde{w} = \sqrt{K\tan K}$

Two new Bleistein sequences are defined, and the Green function for $R/T > 1$ is derived up to second order as:

$$G(M,M',T) = \frac{2^{1/3}\pi}{T^{2/3}\sqrt{a}}[\tilde{E}_0 A_i^2 + \tilde{C}_0 T^{-2/3} A_i'^2 + 2T^{-4/3}\tilde{A}_1 A_i A_i'] \quad (14)$$

The argument of the Airy functions and its derivatives is now $\xi_0^2 T^{2/3}$.

In the vicinity of the front ($a = 1$), the Green function was evaluated by expressions (9) and (14) in order to check the matching at the limit $a = 1$. The difference between the numerical results never exceeded 10^{-6} in all the cases we have tested.

NUMERICAL RESULTS.

We give herein the results for three "typical" configurations: $R = 0$, $R = 10$ and $R = 70$ (*Figures 1, 2 & 3*). The reference curves were computed by inverse Fourier transforming the related frequency domain Green function. The difference between our numerical results and the reference is expressed (*Figures a, b & c*) as a percentage of the Green function's maximum value over the time interval. This percentage never exceed 0.38% for the three considered cases. This agreement between the three methods seems satisfactory for the intended numerical applications.

CONCLUSION.

In this paper, we give two methods in order to evaluate the Green function in the case of finite depth and large time. By associating these formulations with the series method developed for small time range (Clément & Mas, 1994), we are now able to evaluate the function whatever the time and the geometry of the domain. The next step will be the study of the gradient of the Green function in order to resolve seekeeping problems in case of finite depth, in the time domain.

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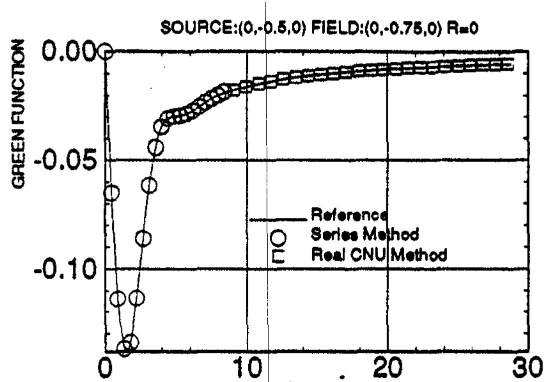

Figure 1

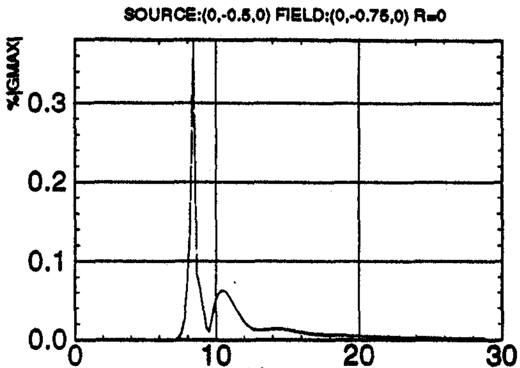

Figure a

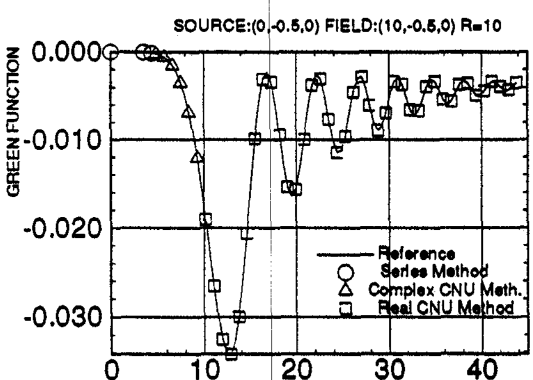

Figure 2

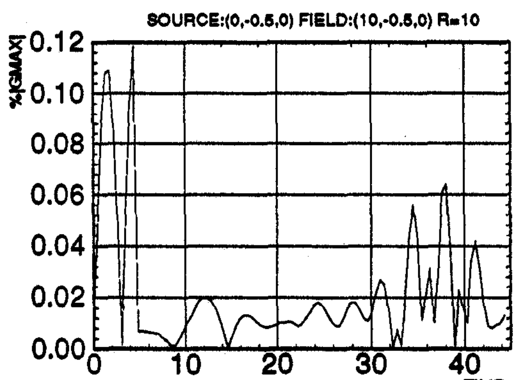

Figure b

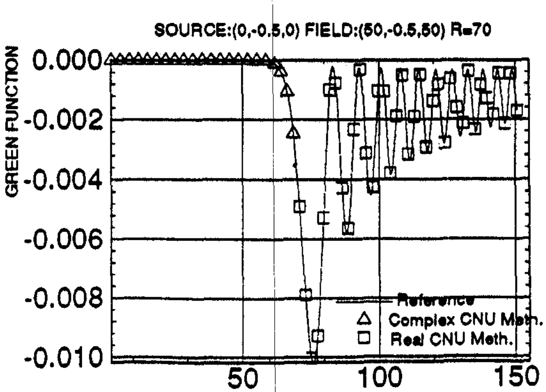

Figure 3

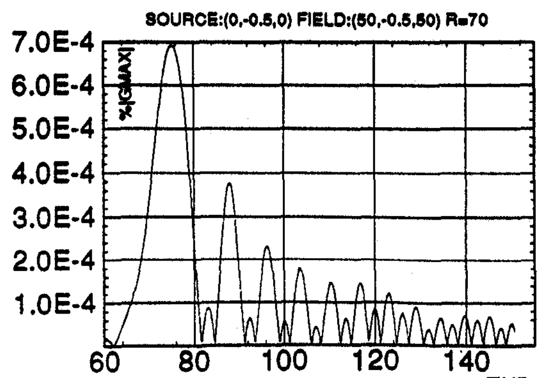

Figure c

DISCUSSION

Doutreleau, Y.: Did you try to use the usual method of steepest descent instead of the CNU method, (because the former is simpler)?.

Mas, S. & Clément, A.: No, because in our method, we don't calculate explicitly the double integral but we simply want to evaluate its asymptotic expansion up to the second order.

Newman, J. N. : I have found that the "second order" correction involving E_2, C_2 and A_3 is required to get accurate results, but that the resulting algorithms are not effective when (say) $R/T < 1/2$ or $R/T > 3/2$. However I am evaluating E_n, C_n, A_n by a least-squares procedure which is ad hoc. Have you found a satisfactory set of algorithms to evaluate E_2, C_2 and A_3?

Mas, S. & Clément, A.: The current development of our work is the evaluation of the gradient of the function. To have the same accuracy as for the Green function, we must evaluate the second order correction. For that, we evaluate analytically the two coefficients E_2 and C_2 owing to the values of the function G^* and its derivatives (up to the fourth derivative). But we did not evaluate A_3 which would correspond, for us, to a third order correction.