Another look at wide-spacing approximations for three-dimensional multiple-scattering problems

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1 Introduction

When a wave meets an immersed body, it is scattered. The scattered field can be calculated in various well-known ways, such as by an integral-equation method.

If there are several bodies, the field scattered from one body will induce further scattered fields from all the other bodies, which will induce further scattered fields from all the other bodies, and so on. This recursive way of thinking about how to calculate the total field leads to the notion of *multiple scattering*; it can be used to actually compute the total scattered field – each step is called an *order of scattering*. Heaviside [2, p. 323] gave a clear qualitative description of this process in 1893.

The simplest approximation, called *single scattering*, is to ignore multiple scattering completely, so that the total scattered field is just the sum of the fields scattered by the individual bodies, each of which is acted on by the incident wave in isolation from the other bodies. This approximation is used widely, but it is only expected to be valid in certain circumstances. Thus, let a be a length-scale characterising the size of the bodies, let $2\pi/K$ be the wavelength of the incident waves, and let b be a length-scale for the spacing between the bodies. Then, the single-scattering approximation is only expected to be appropriate if

$$a/b \ll 1 \qquad \text{and} \qquad Kb \gg 1. \tag{1}$$

The exact multiple-scattering problem is easily formulated: it is an exterior boundary-value problem (with a radiation condition at infinity) where the boundary is not simply-connected. Suppose that the boundary has just two components, S_1 and S_2, and set $S = S_1 \cup S_2$. Then, it is straightforward to reduce the boundary-value problem to a boundary integral equation over S. Computationally, this direct approach can be expensive, especially for problems involving many three-dimensional obstacles. Thus, the goal of the various approximate theories of multiple scattering is to solve the multi-body problem, assuming that we know everything about the scattering of plane waves by each obstacle in isolation. Wide-spacing approximations yield such a theory: we discuss them further.

2 Two dimensions

Consider two immersed bodies; a prototype is a pair of half-immersed, fixed, horizontal, parallel circular cylinders. A regular wavetrain is incident from the left, say, inducing a reflected wave and a transmitted wave from the left-hand cylinder; these waves are not attenuated. In addition, there is a 'local field' near the cylinder. This field decays with distance from the cylinder; in fact, it decays like r^{-2} along the free surface. So, if the right-hand cylinder is sufficiently far away, the local field can be ignored, leaving only the wave transmitted from the left-hand cylinder. This wave will be partly reflected and partly transmitted by the right-hand cylinder, again with an associated local field. Thus, we can envisage waves bouncing back and forth along the free surface between the two cylinders. If we assume that we know the reflection and transmission coefficients for each cylinder in isolation, and for waves incident from the left and from the right, we can match the propagating waves so as to derive approximations to the two-body reflection and transmission coefficients. For a detailed derivation, see Srokosz & Evans [6, §5.3] or Martin [4].

The description above gives the *two-dimensional wide-spacing approximation.* It is essentially one-dimensional; all that matters are the waves propagating along the free surface. Note that the notion of 'orders of scattering' is not appropriate; in effect, we have taken account of *all* orders, with respect to the propagating waves (this is necessary, because these waves do not attenuate). As a consequence, the wide-spacing approximation is known to be extremely effective and accurate (even when the conditions (1) are violated). Indeed, it can be shown that the wide-spacing approximation can be derived rationally by making appropriate asymptotic approximations in a rigorous exact formulation, namely the null-field/T-matrix formulation [4].

3 Three dimensions: overview

Consider two immersed three-dimensional bodies; a prototype is a pair of fixed, half-immersed spheres. Choose Cartesian coordinates $Oxyz$ with the mean free surface in the xy-plane, z increasing with depth. Take the incident wave as

$$\phi_{\text{inc}} = \mathrm{e}^{-Kz}\,\mathrm{e}^{\mathrm{i}K(x\cos\alpha + y\sin\alpha)} = \mathrm{e}^{-Kz+\mathrm{i}Kr\cos(\theta-\alpha)}, \tag{2}$$

where $K = \omega^2/g$ is the wavenumber, α is the angle of incidence and (r, θ) are plane polar coordinates in the xy-plane. Choose an origin O_j at $(x, y) = (\xi_j, \eta_j)$ in the vicinity of S_j, and plane polar coordinates (r_j, θ_j) at O_j, so that $x = \xi_j + r_j\cos\theta_j$ and $y = \eta_j + r_j\sin\theta_j$, $j = 1, 2$.

Imagine the incident wave encountering the first body, S_1; a diagrammatic representation of the scattering process is given in Fig. 1 (a similar process is initiated by the incident wave encountering S_2). The incident wave will induce a scattered wave, spreading out in all directions: it behaves like

$$r_1^{-1/2}\,\mathrm{e}^{\mathrm{i}Kr_1} \qquad \text{as } r_1 \to \infty, \tag{3}$$

on the free surface, and so is attenuated. In addition, there is a local field near the body. This field decays like r_1^{-2} along the free surface. So, if the second body is sufficiently far away, the local field can be ignored, leaving only the attenuated scattered wave from the first body. This wave will be partly scattered to infinity ($\boxed{1}$ in Fig. 1) and partly scattered by the second body, again with an associated local field; this is the *second order of scattering*, $\boxed{2}$. When this wave reaches the first body, it will be of order b^{-1} compared to the incident field; scattering it off the first body gives the *third order of scattering*, $\boxed{3}$.

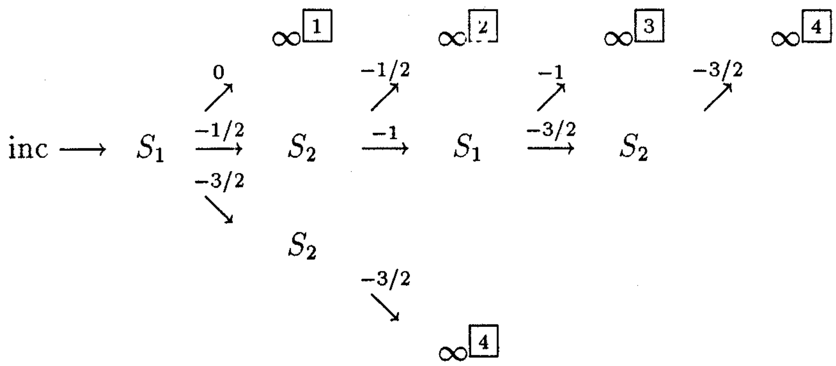

Figure 1: Scattering by two widely-spaced bodies in three dimensions. The number γ above any arrow signifies a contribution of size b^γ, where b is the spacing. Contributions of $O(b^{-2})$ are omitted.

We can make one more step, to fourth order $\boxed{4}$; subsequent steps cannot be justified as they give rise to contributions that are comparable to the neglected local fields.

There is an additional contribution at fourth order. It arises from the next term in (3); specifically, when ϕ_{inc} is scattered by S_1 at O_1, in isolation, the scattered field is given by

$$\phi_{\text{sc}}^1 \simeq I_1 \frac{e^{iKr_1}}{\sqrt{r_1}} \left\{ f_1(\theta_1;\alpha) + \frac{1}{r_1} g_1(\theta_1;\alpha) \right\} \qquad \text{as } r_1 \to \infty,$$

on the free surface, where $I_1 = \exp\{iK(\xi_1 \cos\alpha + \eta_1 \sin\alpha)\}$ is a known phase factor, and f_1 (the *far-field pattern*) and g_1 are assumed known. The term involving g_1 gives a contribution of order $b^{-3/2}$ to the second order of scattering (indicated by the arrows pointing down in Fig. 1), which is comparable to the leading contribution to the fourth order of scattering: both contributions are marked by $\boxed{4}$ in Fig. 1.

f_1 and g_1 come from solving the single-body problem (in fact, g_1 can be expressed in terms of f_1; see (5) below). $f_1(\theta;\alpha)$ gives the far-field amplitude in the direction θ for the incident wave (2) when S_1 is located at O. We assume that $f_1(\theta;\alpha)$ is known (or computable) for any choices of θ and α.

4 Three dimensions: calculations

The approach described above has been used by Greenhow [1] for a number of hemispheres. For two hemispheres, he calculated the second order of scattering $\boxed{2}$, but was 'anxious to avoid' calculating to third order [1, p. 299]! In fact, it is not too difficult to calculate *all* the contributions in Fig. 1. The key to making these calculations tractable was given by Zitron & Karp [7] in their little-cited paper from 1961 on the multiple scattering of acoustic waves in two dimensions.

Let $(\xi_2 - \xi_1, \eta_2 - \eta_1) = b(\cos\delta, \sin\delta)$. Expand ϕ_{sc}^1 in the neighbourhood of O_2, for large b. Straightforward calculations show that

$$\begin{aligned} \phi_{\text{sc}}^1 &\simeq I_1 \, e^{iK(b+X)} b^{-1/2} \left\{ f_1(\delta;\alpha) + b^{-1} \Lambda(X,Y) \right\}, \qquad (4) \\ \Lambda(X,Y) &= \tfrac{1}{2}(iKY^2 - X) f_1(\delta;\alpha) + Y f_1'(\delta;\alpha) + g_1(\delta;\alpha), \end{aligned}$$

where $X = r_2 \cos(\theta_2 - \delta)$, $Y = r_2 \sin(\theta_2 - \delta)$ and $f_1'(\delta; \alpha) = (\partial/\partial\theta) f_1(\theta; \alpha)$ evaluated at $\theta = \delta$; g_1 is given exactly by

$$2\mathrm{i}K g_1(\delta;\alpha) = \tfrac{1}{4} f_1(\delta;\alpha) + f_1''(\delta;\alpha). \tag{5}$$

The expression (4) shows that, to leading order in b, the field scattered by S_1 is approximately a plane wave at S_2, propagating in the direction from O_1 to O_2 and with a known amplitude. Consequently, the scattering of this field by S_2 can be calculated. Moreover, its form gives some justification to Simon's plane-wave approximation [5].

The term involving $\Lambda(X, Y)$ in (4) is not a plane wave, but it can be written as a linear combination of certain derivatives of plane waves with respect to the angle of incidence. Thus, if

$$v(\varphi) = \exp\{\mathrm{i}K r_2 \cos(\theta_2 - \varphi)\},$$

then $v'(\delta) = \mathrm{i}KY\,\mathrm{e}^{\mathrm{i}KX}$ and $v''(\delta) = \mathrm{i}K(\mathrm{i}KY^2 - X)\,\mathrm{e}^{\mathrm{i}KX}$, which are precisely the combinations occurring in (4).

5 Discussion

Complicated three-dimensional multiple-scattering problems can be solved exactly; a powerful method involves using a T-matrix for each obstacle together with an appropriate addition theorem [4], [3]. Nevertheless, there is a place for wide-spacing approximations and, moreover, these approximations are simpler to obtain than is generally believed nowadays. Further work is needed to justify these approximations (asymptotically) and to compare them with exact numerical solutions.

References

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DISCUSSION

McIver, M. What is the error involved in approximating the cylindrical wave by a plane wave?

Martin, P. A.: It is given by equation (4) as $0(b^{-3/2})$. Note that the error consists of three parts (making up Λ), all of which are computable.

McIver, M.: In the 2D problem, the waves are not attenuated. However, after the wave has been scattered several times I would have thought that the amplitude of the wave might be sufficiently reduced that further scattering could be neglected.

Martin, P. A.: To leading order in b (namely b^0), all orders of scattering contribute. However, the numerical contribution of each order of scattering does decrease with increasing order; in this sense you are right. For example, consider two identical bodies, each with a vertical line of symmetry. Locate one at $x = 0$ and one at $x = b$. When a regular wavetrain from $x = -\infty$ is incident, the wide-spacing approximation give the reflection coefficient as

$$R = r\{1 - (r^2 - t^2)e^{2iKb}\}/\Delta$$

where r and t are the complex reflection and transmission coefficients, respectively, of each body in isolation and

$$\Delta = 1 - r^2 e^{2iKb}.$$

Clearly

$$\Delta^{-1} = \sum_{n=0}^{\infty} \left(r^2 e^{2iKb}\right)^n;$$

each term in the series can be interpreted as an order of scattering. Numerically, we get an approximation to Δ^{-1} by truncating the series.