Edge waves along periodic coastlines

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1 Introduction

In a recent paper, to be denoted subsequently by I, Evans & Linton [3] using classical linear water wave theory, proved the existence of edge waves travelling along a periodic coastline consisting of a straight and vertical cliff face from which protruded an infinite number of identical thin barriers, each extending throughout the water depth, provided the barriers were sufficiently long. Because the depth dependence could be separated out, the problem reduced to the solution of the two-dimensional Helmholtz equation, and was identical to an appropriate problem in linear acoustics, optics or electromagnetism involving a 'comb-like' diffraction grating.

Such edge waves are common in classical linear water wave theory but only when the bottom topography is non-uniform. The simplest such solution is that found by Stokes [5] for a uniformly sloping beach. The solution, in the form of a single exponential term, was generalised by Ursell [6] who showed that more and more edge wave modes were possible as the beach slope tends to zero. It is known that edge waves exist whenever a shallow region is joined to a deeper region offshore and Jones [4] proved that at least one such mode exists in this situation.

If the depth of the fluid is constant everywhere it is not obvious that edge waves can exist. For example the only solution in a region of constant depth bounded by a vertical impervious cliff, for waves propagating in the direction of the cliff, is a simple plane wave which does not decay in the direction normal to the cliff, and is not an edge wave. However, as shown in I, if there exists an infinite set of equally-spaced identical thin vertical impervious barriers extending outwards in a direction normal to the cliff and throughout the water depth, edge waves do exist.

A special case of these progressing edge waves described in I is that of standing edge waves and, by symmetry, the problem reduces in this case to a thin barrier, protruding from a vertical wall and mid-way between two parallel vertical walls extending out to infinity. Neumann conditions are to be satisfied on the barrier and the walls and a Dirichlet condition, corresponding to anti-symmetric standing waves, is to be satified on the extension of the barrier out to infinity. This latter condition ensures that a cut-off frequency exists and enables standing edge waves or trapped modes to be constructed as described in Evans [1]. Previously Evans & Linton [2] have used the method of matched eigenfunction expansions to show numerically that such trapped modes occurred when the barrier was replaced by a rectangular block, symmetric about the center-line.

In the present paper we utilise this method to generalise the standing edge waves in the case of the block to progressing waves along a periodic array of rectangular blocks. Although falling short of a rigorous proof of existence the method provides convincing numerical evidence for surface or edge waves travelling along the grating or, in the water-wave context, along the cliff face.

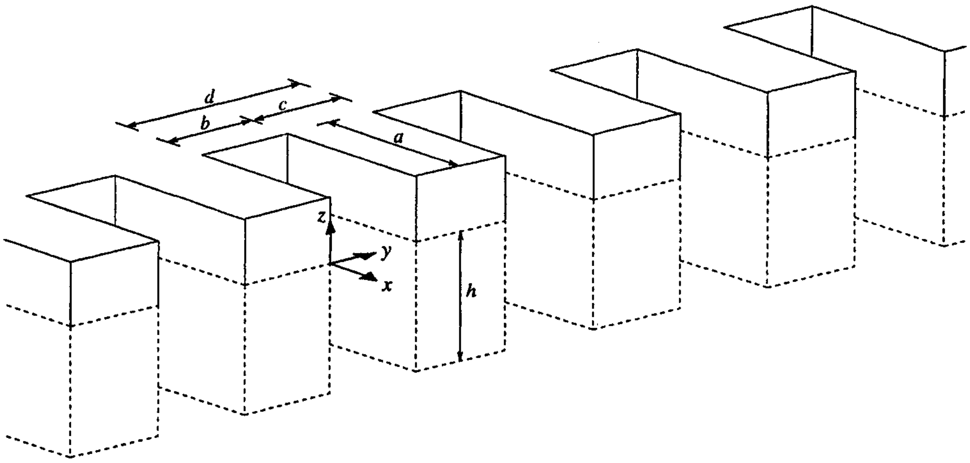

Figure 1: Coastline consisting of a periodic rectangular array.

2 Formulation and Description of Solution

Cartesian coordinates are chosen and the dimensions of the blocks is illustrated in Figure 1. Because they extend throughout the water depth we can write the harmonic velocity potential Φ in the from

$$\Phi(x,y,z,t) = \Re\left\{\phi(x,y)\cosh k(z+h)\,\mathrm{e}^{-\mathrm{i}\omega t}\right\}, \tag{1}$$

where h is the water depth, ω the assumed radian frequency of the edge waves and k is the real positive root of $\omega^2 = gk\tanh kh$. In the context of acoustics this is replaced by $\omega^2 = kc_v$, where c_v is the velocity of sound.

On the basis of either linear acoustics or water waves, we seek a non-trivial $\phi(x,y)$ satisfying

$$\left(\nabla^2 + k^2\right)\phi = 0 \tag{2}$$

in the fluid,

$$\frac{\partial \phi}{\partial n} = 0 \tag{3}$$

on all rigid boundaries, and

$$\phi \to 0, \quad x \to \infty, \quad \text{for all} \quad y. \tag{4}$$

It can be shown that a solution $\phi(x,y)$ to these equations can be constructed for certain values of the parameters and that the corresponding surface elevation in $x > 0$ can be written

$$\eta(x,y,t) = \sum_{n=-\infty}^{\infty} Q_n\,\mathrm{e}^{-\gamma_n x}\cos\left\{\beta_n\left(y - \frac{b}{2}\right) - \omega t\right\} \tag{5}$$

where Q_n is real and is given by

$$Q_n = \frac{1}{\gamma_n d}\left(\frac{2}{\beta_n b}\right)^{\frac{1}{6}} \sum_{m=0}^{N} a_m J_{m+\frac{1}{6}}\left(\frac{\beta_n b}{2}\right) \tag{6}$$

and where a_m satisfies

$$\sum_{n=0}^{N} a_n K_{mn} = \frac{6}{2^{\frac{1}{6}}\Gamma(\frac{1}{6})}\delta_{m0}, \quad m = 0,1,2,\ldots \tag{7}$$

where

$$K_{mn} = \sum_{r=1}^{\infty} P_{rmn} \frac{\coth \alpha_r a}{\alpha_r b} \left(\frac{2}{\pi r}\right)^{\frac{1}{3}} J_{m+\frac{1}{6}}\left(\frac{r\pi}{2}\right) J_{n+\frac{1}{6}}\left(\frac{r\pi}{2}\right) + \sum_{r=-\infty}^{\infty} \frac{1}{\gamma_r d} \left(\frac{2}{\beta_r b}\right)^{\frac{1}{3}} J_{m+\frac{1}{6}}\left(\frac{\beta_r b}{2}\right) J_{n+\frac{1}{6}}\left(\frac{\beta_r b}{2}\right) \tag{8}$$

and where

$$P_{rmn} = \frac{1}{2}\{(-1)^r + (-1)^m\}\{(-1)^r + (-1)^n\}, \tag{9}$$

$$\beta_n = \beta + \frac{2n\pi}{d}, \quad \gamma_n = \left(\beta_n^2 - k^2\right)^{\frac{1}{2}}, \quad \alpha_n = \left(\left(\frac{n\pi}{b}\right)^2 - k^2\right)^{\frac{1}{2}}. \tag{10}$$

For given a/d, b/d, this solution only exists for a particular relation between kd, and hence the wave frequency, and βd the fundamental wavenumber of the edge waves solution. This relationship is obtained from solving

$$kb \tan ka = \frac{6a_0}{2^{\frac{1}{6}}\Gamma(\frac{1}{6})} \tag{11}$$

3 Results

Figure 2 shows the variation of kd for the edge wave solution with a/d for fixed $b/d = 0.4$ and various βd. As a/d increases more modes occur and the difference between successive values approaches π/kd for a given βd.

In Figure 3 the normalised amplitudes of each mode $|Q_n/Q_0|$ are plotted against their wavelengths $\lambda_n/d = 2\pi/\beta_n d$ for the particular set of values $a/d = 1$, $b/d = 0.5$, $\beta d = 2$ for which we find the computed wavenumber $kd = 1.1681$. We see on a log scale how the fundamental mode of wavelength πd dominates, the amplitudes of the higher modes with the exception of Q_{-1} being at least an order of magnitude smaller. In Figure 4 we see that the overall surface elevation at $t = 0$, $x = 0$ of this edge wave is aperiodic but is dominated by the fundamental mode and also appears to affected by a lower 'beat' frequency. Further results plus an indication of how the results (5)–(11) were obtained will be given at the Workshop.

References

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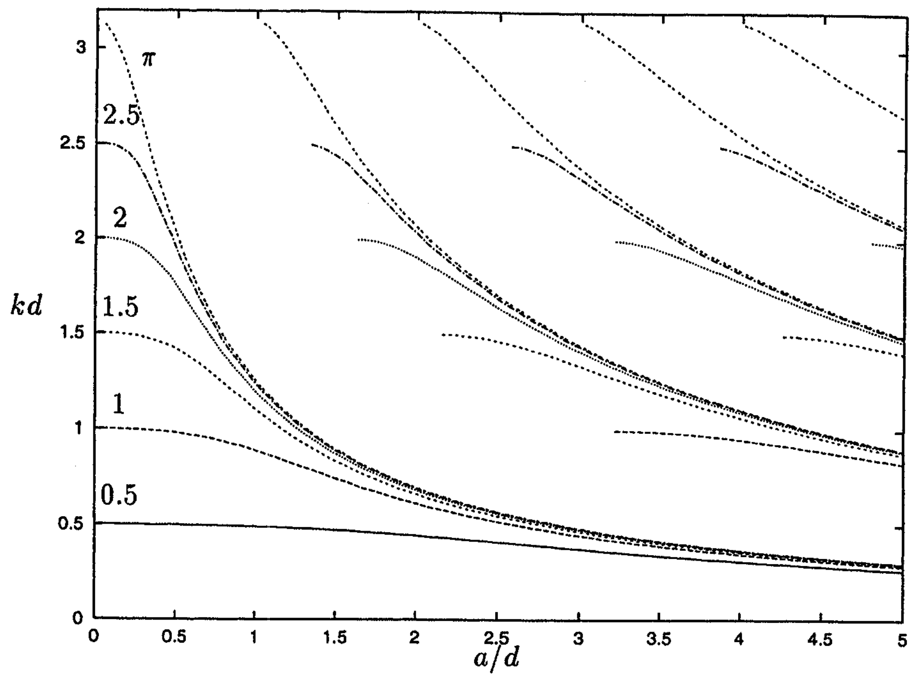

Figure 2: Plots of kd against a/d for various values of βd where $b/d = 0.4$.

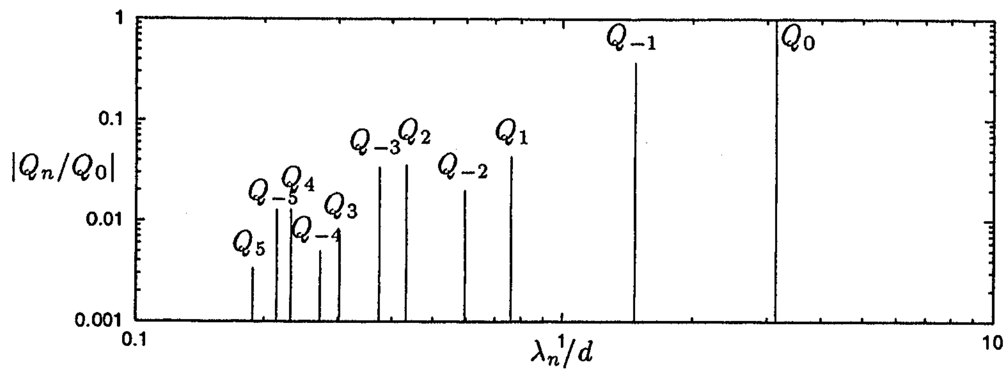

Figure 3: Normalised amplitudes of modes $|Q_n/Q_0|$ against their wavelengths $\lambda_n/d = 2\pi/|\beta d + 2n\pi|$ for the case $a/d = 1$, $b/d = 0.5$, $\beta d = 2$ with the corresponding wavenumber $kd = 1.1681$.

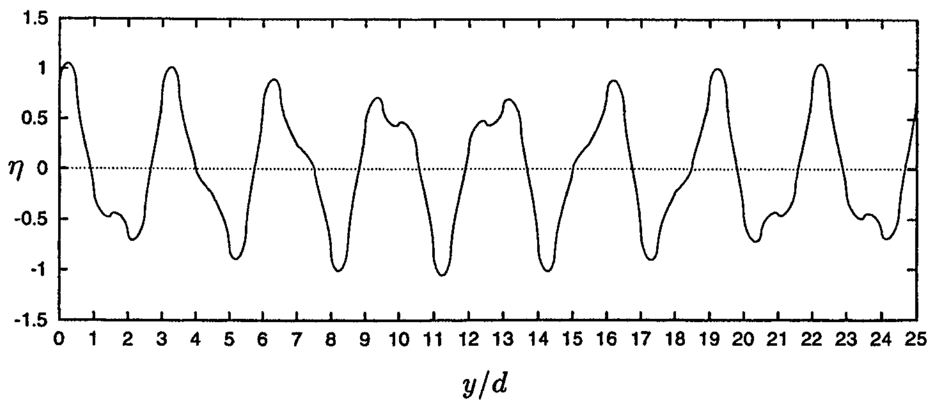

Figure 4: Surface elevation at $x = 0$, $t = 0$ for the case $a/d = 1$, $b/d = 0.5$, $\beta d = 2$ with the corresponding wavenumber $kd = 1.1681$.