Resonances for the three-dimensional Neumann-Kelvin problem in the case of an immersed body

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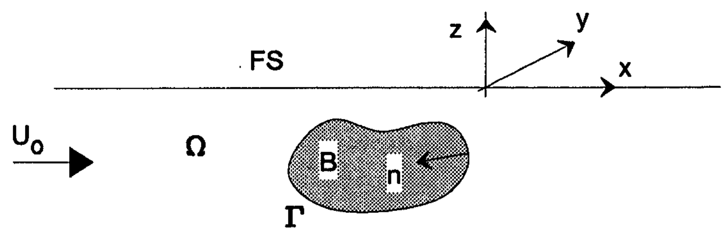
Introduction

We deal with the wave-resistance problem, which consists in determining the strength exerted on a body moving in a vicinity of the free-surface of a fluid at rest. Experimental curves of the wave-resistance with respect to the velocity of the body may have oscillations. We'd like to explain this phenomenon by the existence of **resonances**, i.e., complex values of the velocity for which the potential of the flow becomes infinite. Such a result has been obtained in the two-dimensional case by J-M. Quenez et C. Hazard [1]. We extend it to the three-dimensional case for an immersed body.

1 Neumann-Kelvin Problem

1.1 Equations

We consider the linearized three-dimensional wave-resistance problem, also called Neumann-Kelvin problem. In the referential of the body, the flow comes from $-\infty$ with a uniform velocity equal to $U_0\vec{x}$. Ω is the fluid domain, FS is the free-surface and the body B, whose boundary is Γ, is assumed immersed.



The perturbation of the uniform flow denoted φ_ν, is the solution of the Neumann-Kelvin problem $\mathcal{NK}_\nu$

$$
\mathcal{NK}_\nu \left\{
\begin{array}{lll}
\text{(a)} & \Delta\varphi_\nu = 0 & \text{in } \Omega, \\
\text{(b)} & \partial_n\varphi_\nu = -U_0\,(\vec{x}\cdot\vec{n}) & \text{on } \Gamma, \\
\text{(c)} & \partial_x^2\varphi_\nu + \nu\,\partial_z\varphi_\nu = 0 & \text{on } FS, \\
\text{(d)} & \left\{ \begin{array}{l} \lim\limits_{z\to-\infty} \varphi_\nu = 0, \\ \lim\limits_{z\to-\infty} \partial_z\varphi_\nu = 0, \end{array} \right. & \\
\text{(e)} & \left\{ \begin{array}{l} \varphi_\nu = O(\dfrac{1}{\sqrt{x^2+y^2}}) \\ \nabla\varphi_\nu = O(\dfrac{1}{\sqrt{x^2+y^2}}) \end{array} \right. & x \to -\infty,\ \forall (y,z).
\end{array}
\right.
$$

In this problem, we set $\nu = \dfrac{g}{U_0^2}$. The study of the resonances leads to consider these equations for any datum on Γ, that is to say we replace the left hand side of equation (b) by any given function f.

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This problem is denoted $\mathcal{NK}_\nu(f)$. The aim of this paper is to extend $\mathcal{NK}_\nu(f)$ to complex values of ν and to show that this extended problem, called $\widetilde{\mathcal{NK}}_\nu(f)$, is well-posed except for a denumerable set of values of ν, called **resonances**. Let us briefly describe the successive steps. We first find a problem, denoted $\widehat{Q}_\nu(f)$, set in a bounded domain, equivalent to $\mathcal{NK}_\nu(f)$ when ν is real (part 2.1). Then, we prove that $\widehat{Q}_\nu(f)$ has a unique solution $\hat{\varphi}_\nu$ for any ν in a certain set denoted $\widetilde{\mathbb{C}} - \mathbb{P}$. The elements of $\mathbb{P}$ will be the **resonances** (part 2.2). Using $\hat{\varphi}_\nu$ we find, for any $\nu \in \widetilde{\mathbb{C}} - \mathbb{P}$, the solution $\tilde{\varphi}_\nu$ of problem $\widetilde{\mathcal{NK}}_\nu(f)$, which extends meromorphically $\mathcal{NK}_\nu(f)$ (part 3). Before we give some details, we need to introduce the associated Green function.

1.2 Neumann-Kelvin Green function

The Neumann-Kelvin Green function, $G_\nu(M,P)$, is the potential created at point P by an immersed source located at point M. This function has been determined by many authors (see for instance [2]). The question of uniqueness of this function has been investigated by C. Guttman [3]. We summarize these results in the

Lemma 1 *The Green function is unique and is given by*

$$\begin{aligned} G_\nu(M,P) = & -\frac{1}{4\pi}\left(\frac{1}{r}-\frac{1}{r'}\right) - \frac{\nu}{2\pi}\int_{-\infty}^{+\infty}(\sin \nu S(t,P,M))e^{\nu(1+t^2)(z_P+z_M)}dt \\ & +\frac{\nu}{2\pi^2}\int_{-\infty}^{+\infty}\int_0^{+\infty}(\cos\rho S(t,P,M))PV\left(\frac{e^{\rho(1+t^2)(z_P+z_M)}}{\rho-\nu}\right)d\rho dt, \end{aligned}$$

where $S(t,P,M) = \sqrt{1+t^2}\left((x_P - x_M) + t\,(y_P - y_M)\right)$, *$r$ is the distance between P and M, r' is the distance between P and M', and M' is the symmetrical point of M with respect to the free-surface FS.*

To study resonances, we have to consider complex values of the parameter ν. We need the following lemma, whose proof is too long to be given here:

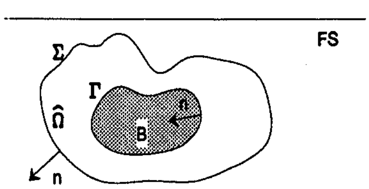
Lemma 2 *Let* $\widetilde{\mathbb{C}} = \{\nu \in \mathbb{C},\ \mathfrak{Re}(\nu) > 0\}$ *and $\mathcal{K}$ a compact domain of*
$\left\{(M,P) \in \left\{\mathbb{R}^2 \times \mathbb{R}^-\right\}^2,\ M \neq P,\ \{z_M = z_P = 0 \text{ and } x_P < x_M \Rightarrow y_M \neq y_P\}\right\}.$
$G_\nu(M,P)$ can be analytically extended with respect to $\nu \in \widetilde{\mathbb{C}}$ in the space of functions $C^\infty(\mathcal{K})$.

This lemma means there exists a function $\widetilde{G}_\nu(M,P)$ analytic with respect to $\nu \in \widetilde{\mathbb{C}}$, uniformly with respect to (M,P) in $\mathcal{K}$, such that for $\nu \in \mathbb{R}^{*+}$, $\widetilde{G}_\nu(M,P) = G_\nu(M,P)$.

2 Reduction to a bounded domain

2.1 Problem $\widehat{Q}_\nu(f)$

We show that we can find a problem set in a bounded domain equivalent to problem $\mathcal{NK}_\nu(f)$, and this will allow us to introduce the resonances. Let Σ be any surface surrounding the body B and not intersecting the free-surface FS.



For $\nu \in \widetilde{\mathbb{C}}$, we seek $\hat{\varphi}_\nu$ solution of the problem $\widehat{Q}_\nu(f)$

$$\widehat{Q}_\nu(f)\begin{cases} \text{(a)} \quad \Delta\hat{\varphi}_\nu = 0 & \text{in } \widehat{\Omega}, \\ \text{(b)} \quad \partial_n\hat{\varphi}_\nu = f & \text{on } \Gamma, \\ \text{(c)} \quad \hat{\varphi}_\nu = \int_\Gamma \left(\hat{\varphi}_\nu(P)\partial_{n_P}\widetilde{G}_\nu(M,P) - f(P)\widetilde{G}_\nu(M,P)\right)d\Gamma_P & \text{on } \Sigma. \end{cases}$$

We now use the method of coupling between variational formulation and integral representation (see Jami-Lenoir [4]). Using the uniqueness of the Green function (lemma 1), we show that for $\nu \in \mathbb{R}^{*+}$, $\mathcal{NK}_\nu(f)$ and $\widehat{Q}_\nu(f)$ are equivalent in the following way

Proposition 1 *For $\nu \in \mathbb{R}^{*+}$,*

if φ_ν is solution of $\mathcal{NK}_\nu(f)$ then $\hat{\varphi}_\nu = \varphi_\nu|_{\widehat{\Omega}}$ is solution of $\widehat{Q}_\nu(f)$, where $\varphi_\nu|_{\widehat{\Omega}}$ is the restriction of φ_ν to $\widehat{\Omega}$,

if $\hat{\varphi}_\nu$ is solution of $\widehat{Q}_\nu(f)$ then $\varphi_\nu(M) = \int_\Gamma (\hat{\varphi}_\nu(P)\partial_{n_P} G_\nu(M,P) - f(P)G_\nu(M,P))\, d\Gamma_P$ is solution of $\mathcal{NK}_\nu(f)$ and $\varphi_\nu|_{\widehat{\Omega}} = \hat{\varphi}_\nu$.

2.2 Study of problem $\widehat{Q}_\nu(f)$

For $\nu \in \widetilde{\mathbb{C}}$, we call $\widehat{\mathcal{R}}(\nu)$ the solution operator of $\widehat{Q}_\nu(f)$, which maps the datum f onto $\hat{\varphi}_\nu$, where $\hat{\varphi}_\nu$ is the solution of $\widehat{Q}_\nu(f)$ when this problem is well-posed. Using a theorem due to S. Steinberg [5], we show the

Proposition 2 $\exists \mathbb{P}$ *discrete countable set of $\widetilde{\mathbb{C}}$ such that $\widehat{\mathcal{R}}(\nu) : f \to \hat{\varphi}_\nu$ is a meromorphic operator with respect to $\nu \in \widetilde{\mathbb{C}}$, whose poles are the elements of $\mathbb{P}$.*

This result means that problem $\widehat{Q}_\nu(f)$ is well-posed when $\nu \in \widetilde{\mathbb{C}} - \mathbb{P}$, and its solution operator $\widehat{\mathcal{R}}(\nu)$ is analytic with respect to $\nu \in \widetilde{\mathbb{C}} - \mathbb{P}$. Moreover, for $\tilde{\nu} \in \mathbb{P}$, $\widehat{\mathcal{R}}(\nu)$ has a Laurent's series expansion in a vicinity of $\tilde{\nu}$. We now give a sketch of the proof of proposition 2 (for more details see [6]). In a first step, we show that solving $\widehat{Q}_\nu(f)$ is equivalent to prove that an operator denoted $\widehat{\mathcal{J}}(\nu)$, is invertible. In a second step, using lemma 2, we show that $\widehat{\mathcal{J}}(\nu)$ form an analytic family with respect to $\nu \in \widetilde{\mathbb{C}}$, of Fredholm operators. We now use Steinberg theorem for such a family: we know that if $\exists \nu_0 \in \widetilde{\mathbb{C}}$ such that $\widehat{\mathcal{J}}(\nu_0)$ is invertible, then $\exists \mathbb{P}$ discrete countable set of $\widetilde{\mathbb{C}}$ such that, $\forall \nu \in \widetilde{\mathbb{C}} - \mathbb{P}$, $\widehat{\mathcal{J}}(\nu)$ is invertible and its inverse, that is $\widehat{\mathcal{R}}(\nu)$, is a meromorphic operator whose poles are the elements of $\mathbb{P}$. To show that $\widehat{\mathcal{J}}(\nu_0)$ is invertible, we follow Vainberg and Maz'ya (see [7]) and prove that there is one and only one solution to $\mathcal{NK}_{\nu_0}(f)$, when ν_0 is closed to 0^+. Then, using the equivalence between $\mathcal{NK}_{\nu_0}(f)$, and $\widehat{Q}_{\nu_0}(f)$ (see proposition 1), we see that $\widehat{\mathcal{J}}(\nu_0)$ is injective and also invertible thanks to Fredholm's alternative.

3 Meromorphic continuation of Neumann-Kelvin problem

3.1 Solution and continuation operators

For $\nu \in \mathbb{R}^{*+}$, we call the solution operator of $\mathcal{NK}_\nu(f)$, $\mathcal{R}(\nu)$, which maps the datum f onto φ_ν, where φ_ν is the solution of $\mathcal{NK}_\nu(f)$, when this problem is well-posed. For $\nu \in \widetilde{\mathbb{C}} - \mathbb{P}$, we call continuation operator, $\widetilde{\mathcal{R}}(\nu)$, which maps the datum f onto $\tilde{\varphi}_\nu$, where $\tilde{\varphi}_\nu$ is defined by
$\forall M \in \Omega,\ \tilde{\varphi}_\nu(M) = \int_\Gamma \left(\hat{\varphi}_\nu(P)\partial_{n_P}\widetilde{G}_\nu(M,P) - f(P)\widetilde{G}_\nu(M,P)\right) d\Gamma_P$ and $\hat{\varphi}_\nu = \widehat{\mathcal{R}}(\nu)(f)$ is the unique solution of $\widehat{Q}_\nu$.

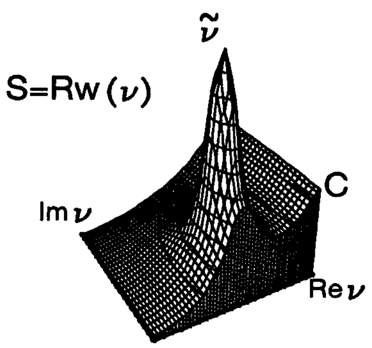
3.2 Resonances

First, using proposition 2, we prove that $\tilde{\varphi}_\nu$ is a meromorphic function with respect to $\nu \in \widetilde{\mathbb{C}}$. Then, thanks to proposition 1, we show that for $\nu \in \mathbb{R}^{*+} - \mathbb{P}$, $\tilde{\varphi}_\nu = \varphi_\nu$, and $\tilde{\varphi}_\nu$ is also the meromorphic continuation of φ_ν. Thus $\widetilde{\mathcal{R}}(\nu)$ is the meromorphic continuation of $\mathcal{R}(\nu)$ with respect to $\nu \in \widetilde{\mathbb{C}}$ and we can show that its poles are the elements of $\mathbb{P}$. Moreover, using the same techniques as in proposition 1, we prove that $\widetilde{\mathcal{R}}(\nu)(f)$ is the solution operator of $\widetilde{\mathcal{NK}}_\nu(f)$, where this problem is the same problem as $\mathcal{NK}_\nu(f)$, except that equation (e) is replaced by $\tilde{\varphi}_\nu(M) = \int_\Gamma \left(\tilde{\varphi}_\nu(P)\partial_{n_P}\widetilde{G}_\nu(M,P) - f(P)\widetilde{G}_\nu(M,P)\right) d\Gamma_P$.
The fact that $\widetilde{\mathcal{R}}(\nu)$ coincide with $\mathcal{R}(\nu)$ for $\nu \in \mathbb{R}^{*+}$, shows that $\widetilde{\mathcal{NK}}_\nu(f)$ is equivalent to $\mathcal{NK}_\nu(f)$ for $\nu \in \mathbb{R}^{*+}$, and thus $\widetilde{\mathcal{NK}}_\nu(f)$ is really an extension of $\mathcal{NK}_\nu(f)$. We summarize all these results in the

Theorem 1 *$\widetilde{\mathcal{R}}(\nu)$ is the meromorphic continuation of $\mathcal{R}(\nu)$ (solution operator of $\mathcal{NK}_\nu(f)$), that is the solution operator of $\widetilde{\mathcal{NK}}_\nu(f)$ (extension of $\mathcal{NK}_\nu(f)$). The poles of $\widetilde{\mathcal{R}}(\nu)$ are the elements of $\mathbb{P}$, and we call them* **resonances.**

3.3 Interpretation

For the wave-resistance problem, the datum $f = -U_0\,(\vec{x}\cdot\vec{n})$. Theorem 1 shows that $\tilde{\varphi}_\nu = \widetilde{\mathcal{R}}(\nu)\,(f = -U_0\,(\vec{x}\cdot\vec{n}))$ is a meromorphic continuation of φ_ν. If $\tilde{\nu}$ is one of these poles, this entails that the wave-resistance R_w, that is an integral depending on $\tilde{\varphi}_\nu$, has a Laurent's series expansion in a vicinity of $\tilde{\nu}$.



We see that if $\tilde{\nu}$ is closed to the real axis, the fact that the surface S tends to infinity near $\tilde{\nu}$ influences the shape of the curve C that is the intersection between S and the plane $\mathfrak{Im}\,(\nu) = 0$. This entails the oscillations we notice on wave-resistance curves.

Conclusion

We obtain the result in the case of an immersed body. By using a surface Σ intersecting the free-surface, we can prove the same result by a more complicated way. We intend to use this second method to extend the result to the case of a surface-piercing body.

References

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DISCUSSION

Kuznetsov, N.: The work is very interesting. It is done in the best traditions of the French school. The only remark concerns references. The following paper by Kochin should be mentioned, since results of Maz'ya & Vainberg are essentially based on it.

Kochin, N. *On the wave-making resistance and lift of bodies submerged in water*. Proc. Conf. Theory of Wave Resistance, Moscow, 1937, pp. 65-134. English translation in SNAME Tech. Res. Bull. 1-8 (1951).

Doultreleau, Y. : I'd like to thank Professor Kuznetsov for his comment. First, I have carefully read the paper of Maz'ya and Vainberg, and a part of my talk is essentially based on their ideas. That must be said. Second, I didn't know this work by Kochin and I am grateful to Prof. Kuznetsov for making this remark .