Modified Pressure Impulse Theory for Wave Impact

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1 Introduction

Cooker and Pergrine (1990) used a theory of pressure impulse (Lamb, 1932) to investigate the high, short-lived pressures and sudden changes in the velocity field which occur when a wave hits a sea wall. Although an impact is very brief, lasting only milliseconds, the pressures exerted are at least ten times hydrostatic. Recently Chan (1994) has shown that predictions accord with experiments. However we have not yet justified analytically, the use of pressure impulse P (the time-integral of pressure during the short time of impact), particularly for fluid near a free surface. P is approximately a solution of Laplace's equation and the free-surface boundary condition at the instant of impact is taken to be $P = 0$. Pressure impulse theory rests upon neglecting the nonlinear terms in Euler's equations, but near the free surface $-\partial\phi/\partial t \approx \frac{1}{2}\mathbf{u}^2$ and the nonlinear terms are not relatively small. In this work we take account of these previously neglected terms.

2 Analysis

Instead of using the time-integral of pressure P, we define, at a point $\mathbf{x}$ in a potential flow, the quantity:

$$Q(\mathbf{x}) = \rho(\phi_b - \phi_a) \tag{1}$$

where $\nabla\phi_b = \mathbf{u}_b(\mathbf{x})$ is the given velocity before impact, and $\nabla\phi_a = \mathbf{u}_a(\mathbf{x})$ is the velocity after impact. Throughout we use subscripts $_b$ and $_a$ to denote before and after impact, respectively. In the interior of the fluid domain, Q satifies Laplace's equation exactly, with Neumann conditions on the solid boundaries, as shown in figure 1. The impact on the vertical wall is represented by the condition $\partial Q/\partial n = u_n/\rho$, where u_n is a prescribed velocity component normal to the wall. Also Q is asymptotically equal to P as the impact duration Δt tends to zero. (An impulsive pressure field does not generate vorticity so the existence of the velocity potential ϕ_a is assured if ϕ_b is given.)

Since an impact is a sudden change in velocity we may plausibly write the following construction for the three velocity components:

$$u_i(\mathbf{x}, t) = u_{bi}(\mathbf{x}) + H_i(t)\{u_{ai}(\mathbf{x}) - u_{bi}(\mathbf{x})\}. \tag{2}$$

where i=1,2,3 . For an incompressible fluid $\nabla.\mathbf{u}_b = \nabla.\mathbf{u}_a = \nabla.\mathbf{u} = 0$ and a separation of variables argument leads to $H_1(t) = H_2(t) = H_3(t) = H(t)$, say, and instead of (2)we have

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_b(\mathbf{x}) + H(t)\{\mathbf{u}_a(\mathbf{x}) - \mathbf{u}_b(\mathbf{x})\}. \tag{3}$$

Here $H(t)$ is expected to increase monotonically from the value 0 as $t \to -\infty$ to the value 1 as $t \to +\infty$. The significant change in H (and hence in the velocity) occurs in the short time interval $[t_b, t_a]$, which is more conveniently written $[-\Delta t/2, \Delta t/2]$, where $\Delta t = t_a - t_b$ is the impact duration. Substituting (3) into Bernoulli's law we have an expression for pressure

$$p(\mathbf{x}, t) = p_b(\mathbf{x}) + H(t)\{p_a(\mathbf{x}) - p_b(\mathbf{x})\} + Q\frac{dH}{dt} + \frac{1}{2\rho}[H - H^2](\nabla Q)^2. \tag{4}$$

where $\nabla Q = \rho(\mathbf{u}_b - \mathbf{u}_a)$. The pressure, according to (4) undergoes a change from p_b before impact to p_a after impact, where p_a is the pressure according to Bernoulli's law, and is known once Q is known. During impact the term in (4) with greatest modulus is generally QdH/dt, and is associated with the spike in pressure records long-observed by experimenters. The coefficient of the last term in (4) is $\frac{1}{2}(H - H^2)$, which is 0 before and after impact and is a "spike" during impact with a maximum value of only $\frac{1}{8}$ so we may neglect the last term in (4) in most of the fluid domain. However, near the free surface, where the flow speed squared (directly proportional to $(\nabla Q/\rho)^2$) is greatest, we expect $(8\rho)^{-1}(\nabla Q)^2$ to be a significant pressure compared with QdH/dt. Indeed these terms must balance each other on the free surface itself, where $p = p_b = p_a = 0$. From these considerations we obtain the form of $H(t)$ and an appropriate free-surface boundary condition for Q. A separation of variables argument leads to $2dH/dt/(H - H^2) = k$ with solution

$$H(t) = \frac{1}{1 + e^{-kt/2}} \tag{5}$$

where k is a positive separation constant. If we take $k = 18.4/\Delta t$ (where $t_b = -\Delta t/2$ and $t_a = \Delta t/2$) then $H(t_b) = 0.01$ is conveniently small, and $H(t_a) = 0.99$ is conveniently near to 1, the desired limit of H as $t \to \infty$. Generally k is so large that H resembles Heaviside's function, as shown in figure 2. The other part of the above separation of variables leads to a free-surface boundary condition for Q:

$$Q = \frac{-1}{k\rho}(\nabla Q)^2 \tag{6}$$

If U_0 is a typical speed of impact and h the wave height then in dimensionless (starred) variables the surface boundary condition is $Q^* = -\epsilon(\nabla^* Q^*)^2$ where $\epsilon = U_0\Delta t/18.4h$ is much less than unity. Equation (6) replaces the surface condition that the pressure impulse $P = 0$. In equation (4), away from the free surface, the peak pressure is approximately $Qk/8$ or $2.3Q/\Delta t$ or $2.3Q^*\rho U_0 h$. Recalling that $Q \approx P$ for short-duration impacts this value of the peak pressure agrees well with that of Cooker et al (1990) $2P/\Delta t$, derived from physical arguments.

Although in (6) we have a nonlinear boundary condition, we gain in this approach a treatment of the kinetic energy (though see Korobkin, A. (1995), this meeting, for further analysis).

3 Changes in the total energy brought about by impact

During the short time of impact the free surface moves little and we may neglect changes in the potential energy of the flow. According to pressure impulse theory, the kinetic energy lost by a wave impacting a region S of a rigid impermeable solid surface is

$$\Delta E = \frac{1}{2}\int_S u_n P dS \tag{7}$$

where $u_n > 0$ is the component of wave impact velocity along the outward normal of the domain. In terms of Q the energy loss has an extra contribution from the free surface, F:

$$\Delta E = \frac{1}{2}\int_S u_n Q dS + \frac{1}{2k\rho^2}\int_F (\partial Q/\partial n - 2\rho u_n)(\nabla Q)^2 dF. \tag{8}$$

The first integral in (8) is positive and the second integral is negative, principally because $\partial Q/\partial n < 0$ on F. Hence for an appropriate choice of Δt (i.e. k) we may ensure that $\Delta E = 0$, which is a reasonable expectation for the types of impact computed by Cooker and Peregrine (1991), and by others. This also suggests that the energy lost to the impact of water against a solid surface is gained by fast-moving fluid near the free surface, notably in violent jets.

References

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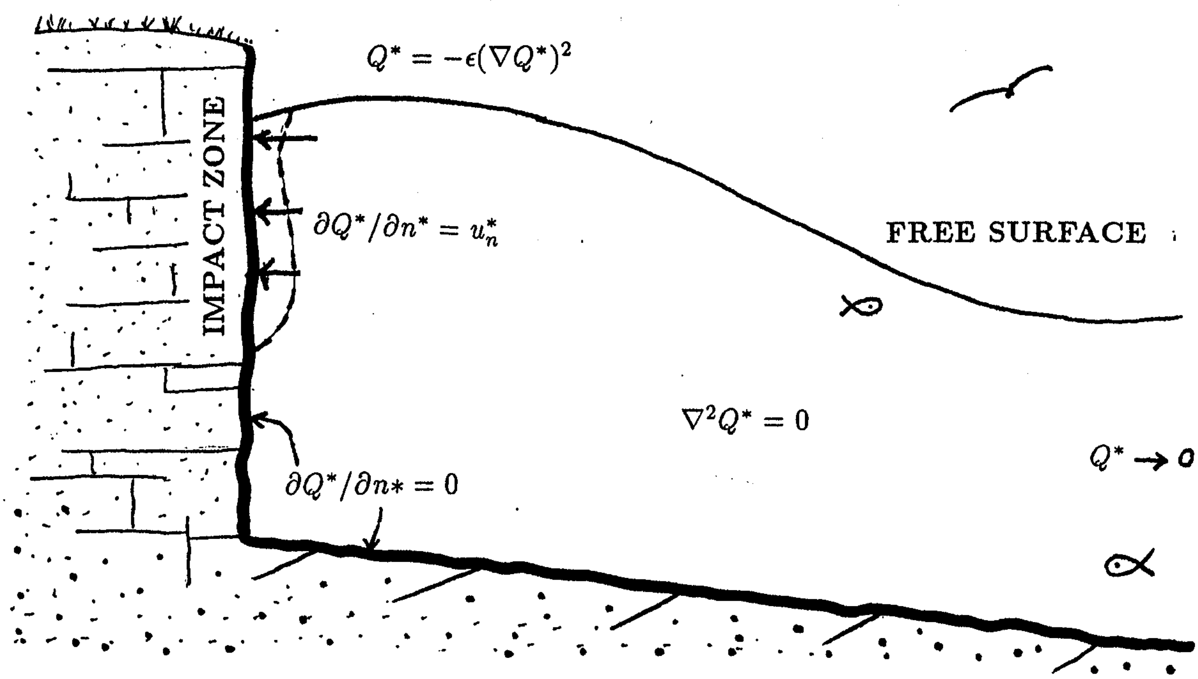

FIGURE 1: Boundary-value problem for $Q^* = Q/(\rho U_0 h)$ in the domain of a wave at the instant of impact against a sea wall, where U_0 is a characteristic impact speed and h is the overall depth. The boundary conditions are the same as for pressure impulse, P, except for the new free-surface condition in which $\epsilon = U_0 \Delta t/(18.4h)$.

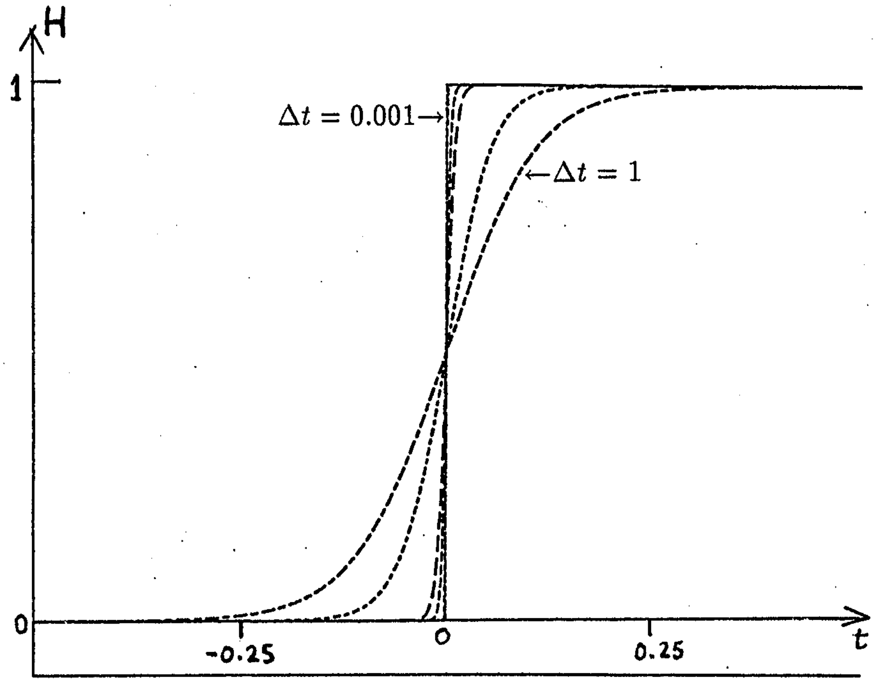

FIGURE 2: The function $H(t)$, which governs the suddenness of the change in velocity field due to the impact, plotted for impacts of various duration
Δt = 1, 0.5, 0.01 0.05, 0.01, 0.005, 0.001 seconds. These times are much shorter than typical coastal wave periods, and the last three are typical durations for wave impact pressures.