Treatment of the dynamics of the intersection of a free-surface and a solid body

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In numerical simulations of the dynamics of water waves and floating bodies, the numerical treatment of the water line on the body has generally caused trouble. This is especially true for the semi-Lagrangian boundary integral methods originally developed by Longuet-Higgins & Cokelet[4, 7]. The difficulties may arise from at least three sources: the intersection between the free-surface and the body is the confluence of two different sorts of boundary condition, a normal velocity condition on the body and an essentially tangential boundary condition of the surface; the presence of thin jets of fluid in high acceleration simulations[2, 3]; and complex wave modes of a wide range of wavenumbers in low acceleration simulations[8].

However, recently successful computations of two-dimensional, extreme standing waves, both their form and stability in both infinite[5] and finite[6] depth water, has shown that we can treat the intersection between a free-surface and a body very simply. Standing waves occur in a container with vertical and immobile walls. In this sense they are a very special problem. Nonetheless, the ease with which we have been able to simulate complex wave dynamics, involving crests which are nearly in free fall, suggests that we should try to adapt these special features to more difficult wave-body simulations. The challenge is to overcome the above mentioned dynamical difficulties with a numerical scheme that is accurate, simple to code, and robust enough to cater for a wide range dynamical behaviour.

1 Numerical approximation of a boundary integral

The standing wave simulations are performed by representing the free-surface as a vortex sheet: at position $z = x + iy$, the complex velocity

$$\bar{q}(z) = u - iv = \frac{1}{2\pi i} \int \frac{a(j)}{z - Z(j)} dj \tag{1}$$

where $a(j)$ is the vorticity density on the surface $Z(j)$, and j is some parameter of the free-surface. Our treatment of this and related integrals is simply the trapezium rule:

$$\bar{q}(z) \approx \frac{1}{2\pi i} \sum_j \frac{a(j)}{z - Z(j)} \, . \tag{2}$$

Although simple, this rule becomes accurate exponentially quickly as z moves away from the free-surface. The error arises from two causes. First there are end effects if we truncate such a sum to a finite number of terms; these end induced errors are located primarily near the ends of the finite vortex sheet that the finite sum approximates. In the standing wave simulations these

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errors are avoided by requiring 2π-periodicity in the horizontal:

$$\bar{q}(z) \approx \frac{1}{4\pi i} \sum_{j=1}^{N} a(j) \cot\left(\frac{z - Z(j)}{2}\right) \tag{3}$$

where the free-surface and the vorticity strength is N-periodic in the parameter j. The second and, we consider, more serious error is understood by recognising that the sum (2) represents physically discrete vortices distributed along the free-surface. This causes the numerically computed velocity field to be "bumpy", like the so-called "cats-eyes", on a scale of the vortex separation. These bumps decay exponentially quickly to a smooth and accurate velocity field as z moves away from the surface. Note that a higher-order integration scheme may well do worse; for example, Simpson's rule with its $\ldots, \frac{2}{3}, \frac{4}{3}, \frac{2}{3}, \frac{4}{3}, \frac{2}{3}, \ldots$ pattern introduces lumpiness of the vorticity on a scale of *twice* the vortex spacing and so introduces errors which penetrate twice as far into the fluid as does the errors of the simple trapezium rule.

Now imagine such distributed discrete vortices representing a free-surface in the vicinity of a solid body which pierces the surface. Both of the above mentioned errors occur. The transition from the surface boundary condition to the body boundary condition on the normal body is an abrupt change which can only reasonably be dealt with by separately treating the integrals over these disparate types of boundaries. Thus end effects inevitably arise in the computation of the contribution of the surface vorticity to the velocity field. The second source of error is disastrous. The bumpiness induced by the discrete vorticity distribution is inevitable on the length-scale of the vortex separation, but the discretisation of the solid body is also going to be on a similar length-scale and so will always feel the bumpiness. Thus the numerically computed velocity field may well have large differences by just small changes in the discretisation—from placing a body point one side or the other of a nearby surface vortex (recall that thin jets of water are likely to be developed in any dynamic simulation of wave-body interactions). Indeed, Davidson[2] found that despite high resolution and all possible care, the vertical bow of an accelerating barge often overtook points of the surface in the jet in front of the barge. Apparently these problems could be reduced acceptably by using a smoother distribution of vorticity on the surface, spline or piecewise linear for example. These certainly reduce the problems and may be one route towards success, but they do not eliminate the problems; the discontinuity of a derivative always introduces "bumps" albeit weaker and weaker for higher order smoothness. Furthermore, such schemes incur a great cost in complexity of coding. And yet the extreme standing wave simulations of Mercer & Roberts[5, 6] had no problems despite using simple discrete vortices. We seek methods which are similarly simple, robust and accurate.

Similar considerations to the above also hold for how the discretisation of the body boundary affects the free surface.

The standing wave was simulated with the wall boundary condition satisfied by distributing equal but opposite image vortices:

$$\bar{q}(z) \approx \frac{1}{4\pi i} \sum_{j=1}^{N/2} \left[a(j) \cot\left(\frac{z - Z(j)}{2}\right) - a(j) \cot\left(\frac{z + \bar{Z}(j)}{2}\right) \right] \tag{4}$$

This ensures that although the vertical velocity on the wall at $x = 0$ may be bumpy, the normal velocity is smooth, namely zero. Thus introducing an image of the free surface within the body may be sufficient. However, in general simulations the surface and the body will not intersect at right angles. Thus in contrast to the standing wave simulations, the image of the surface and the surface itself will not meet smoothly at the intersection with the body; consequently end effect errors will still be large. To avoid end effects we must continue the free-surface smoothly through the intersection point into the interior of the body. Similarly, and more physically, the discretisation of the body must be continued above the surface. The necessary extra equations for the extra vortices and sources may be obtained by smooth continuation through the intersection.

However, such continuation, rather than imaging, results in an induced velocity field which is bumpy on the body, and vice-versa. The challenge, then, is to correct this defect of the velocity field near the surface.

2 Uniformly valid velocity field

That it should be possible to remove the bumps is indicated by the computation of the velocity actually *on* the free surface in order to evolve the surface in time. On the free-surface itself, the discrete nature of the vortex distribution is its most extreme, exhibiting pole singularities. In [7] the singularity at a surface point $Z(k)$ is removed by subtracting an equal and opposite singularity of an integral which is net zero. By some careful limits we then find the appropriate value of the integrand to be used in the discrete vortex sum (2) at the surface point $Z(k)$. The resulting numerical scheme is spectrally accurate.

We now describe a numerical scheme for the evaluation of integrals such as (1) which is uniformly valid throughout the fluid. Consider a field point z which is near to the free-surface. Suppose the nearest discrete vortex of the surface is located at the point $Z(k)$. The discrete vortices distributed on the surface in the neighbourhood of $Z(k)$ are there to represent a continuous vortex sheet. Instead, in the sum (2) they produce a velocity field which is nearly that of a uniform line of vortices in the neighbourhood. We largely cancel the discrete vortices in the neighbourhood by subtracting a vortex array of the local strength $a(k)$, local spacing $|Z'(k)|$, and direction $Z'(k)$, namely

$$\frac{a(k)}{2Z'(k)i}\cot\left(\frac{\pi(z-Z(k))}{Z'(k)}\right) . \tag{5}$$

Subtracting this effectively removes the bumpiness in the local velocity field. This vortex array, especially away from the surface, is interpreted as representing a vortex sheet in the direction of the local surface tangent and of vorticity density $a(k)/|Z'(k)|$. Thus we eliminate its large-distance effects by adding the velocity field due to the corresponding continuous vortex sheet which below the surface is

$$\frac{a(k)}{2Z'(k)} . \tag{6}$$

Close to the surface, in the neighbourhood of $Z(k)$, this vortex sheet velocity field is that induced by the local vorticity $a(k)$ being spread continuously over the surface. It replaces the bumpiness in the local velocity field that has been cancelled by the discrete vortex array. Thus a smooth velocity field may be computed simply from

$$\bar{q}(z) = \frac{1}{2\pi i}\sum_j \frac{a(j)}{z-Z(j)} - \frac{a(k)}{2Z'(k)i}\cot\left(\frac{\pi(z-Z(k))}{Z'(k)}\right) + \frac{a(k)}{2Z'(k)} . \tag{7}$$

Indeed such an expression is uniformly valid throughout the flow field. It may be recognised as being of the form of a uniformly valid expression in matched asymptotic expressions[1, Chapt. 9]:

$$\bar{q}_{\text{uniform}} = \bar{q}_{\text{outer}} - \bar{q}_{\text{intermediate}} + \bar{q}_{\text{inner}} . \tag{8}$$

Here the "outer" approximation is the numerical integral (1) which is spectrally accurate in the field far from the surface, where "far" is on the scale of the point spacing. The "inner" approximation is just the local velocity field (6) induced by the local vortex sheet of strength $a(k)/|Z'(k)|$. The "intermediate" approximation is the local vortex sheet approximated by discrete vortices. Far from the free-surface, the contributions from the intermediate and inner terms cancel, just leaving the accurate outer approximation. Close to the free-surface, the bumps in the outer approximation and the bumps in the intermediate approximation cancel just leaving the accurate inner approximation together with the accurate influences of distant parts of the free-surface expressed through uncancelled parts of the outer approximation. This uniformly valid approximation is being used to investigate the structure of the pressure, velocity and acceleration fields beneath a standing wave[9].

3 Conclusion

Our proposal then for overcoming the major numerical difficulties associated with the intersection between a body and the free-surface is: firstly, to smoothly continue the free-surface into body and the body up into the air in order to avoid end effects in the integrals; secondly, when the body is in the neighbourhood of the free surface the uniformly valid approximation (7) should be used to compute the effect of the surface on the body; thirdly and symmetrically, when the free surface is in the neighbourhood of the body, a similarly uniformly valid approximation should be used for the body's influence on the velocity field at the surface.

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DISCUSSION

Yue D.K.P: First let me express my appreciation for your very insightful observations which I believe are right en-pointe. I have two questions regarding implementation of your ideas in practice:

1) The intersection flow is not necessarily smooth in general so that determining the extended unknowns beyond the body/water surface based on "smoothness" considerations may be inappropriate;

2) Your Eq.(6) relies on the availability of a closed-form sum for an infinite array of constant strength singularities on a straight line. In the situation with a more uneven (or perhaps disparate) grid/singularity, and for 3D where the doubly periodic sum can not be obtained analytically, the present prescription may not be so easy. Perhaps one needs only to subtract out a limited patch of such singularities but then there may be concern about smoothness at the edges of these patches.

Roberts A.J.: 1) If the non-smoothness is numerical, then good numerics should get rid of it. If the non-smoothness is physical, then we have a modelling decision – do we want the numerics to resolve the phenomenon or not. This is a personal dicision. I guess an example is the complex details in front of an instantaneously on easily accelerated plate. One has to decide what level of detail to resolve and then implement the numerical scheme accordingly.

2) Correct. In 2D it is convenient to use an infinite array of vortices on the tangent due to the availability of analytic expressions for both the vortex array, namely a cotangent, and for the continuous corresponding vortex sheet. In principle the velocity fields in any local near the boundary can be made smoothly accurate by subtracting a patch of the discrete singularities and adding the corresponding patch of continuously distributed singularity. The patch only need be big enough so that the end effects of the patch do not affect the smoothness of the velocity fields in the locale region under consideration. However, there is possibly an accuracy issue. To be accurate the fields of the discrete and the continuous patch have to cancel in the far field. Indeed the ideal case is that they cancel exponentially quickly as one moves away from the surface out to the far field – as occurs for the infinite patch. Presumably, for a finite patch the cancellation decays only as an inverse power of distance. A problem with a finite patch is the computation of the smooth velocity field due to the continuous singularity distribution – this could be difficult.

Clement A.: What kind of boundary condition do you write on the extensions of the free surface into the body and the body into the air? How do you take these contributions into account in the Green formula to derive the grobal Boundary Integral Problem?

Roberts A.J.: Both these equations have the same answer. The extension of the free surface into the body and the extension of the body into the air are extrapolations of the actual data on the free surface and on the body in the neighborhood of the intersection point. Thus the extensions do not really appear within the Green formula – only in the numerical approximation to the integrals. In essence the extensions, being extrapolations of the end few points, form a nonlinear weighting scheme of the data at these end few points in the integrals.