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WAVE FORCES ON FLOATING BODIES IN SLOW YAW-MOTION

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Wave drift damping of floating bodies in the three horizontal modes of motion is a current important problem within offshore technology. Earlier works have considered wave drift damping due to translatory motions. In this contribution a theory is developed for evaluating wave drift damping due to a slow yaw-motion, i.e. a slow rotation about the vertical axis. The problem is considered in the relative frame of reference rotating with the slow yaw motion of the body. This frame of reference is connected to a fixed frame of reference by the rotation angle $\alpha(t)$, which is a slowly varying function of time. There is no restrictions on the magnitude of the yaw-angle, thus $\alpha(t) = O(1)$. The angular velocity, $\Omega = \dot{\alpha}$, is assumed to be small, however.

Potential theory is applied to describe the flow and the fluid pressure. An exact expression is developed for the fluid pressure in the relative frame of reference. Perturbation expansions in the wave amplitude and the slow yaw-velocity are then applied to the potentials. The problem is solved to leading order in the wave amplitude and the slow yaw-velocity by means of integral equations. The potential appears as unknown on the wetted body surface only. A discretization of the free surface is needed for ordinary integration, however. The wave drift damping coefficient in the yaw mode is then obtained by conservation of angular momentum.

Relevant to the present problem is a recent work by Newman (1993) who describes the motion from the absolute reference system, assuming that the rotation angle α is small.

EQUATION OF MOTION IN THE RELATIVE FRAME OF REFERENCE

The equation of motion reads

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho\mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p - \rho g\nabla z + \mathbf{H} \tag{1}$$

Here, $\mathbf{v}$ denotes the fluid velocity, ρ the density, p the fluid pressure, g the acceleration of gravity. x, y, z coordinates with the z-axis being vertical upwards. Let $\mathbf{\Omega} = \Omega\mathbf{k}$ ($\Omega = \dot{\alpha}$) denote the slow angular velocity. $\mathbf{H}$ is composed by the Coriolis force, $-2\rho\mathbf{\Omega}\times\mathbf{v}$, the centrifugal force, $-\rho\mathbf{\Omega}\times\mathbf{\Omega}\times\boldsymbol{x}$, and the fictive force due to the angular acceleration, $-\rho\dot{\mathbf{\Omega}}\times\boldsymbol{x}$, where a dot denotes time derivative. Thus,

$$\mathbf{H} = -2\rho\mathbf{\Omega}\times\mathbf{v} - \rho\mathbf{\Omega}\times\mathbf{\Omega}\times\boldsymbol{x} - \rho\dot{\mathbf{\Omega}}\times\boldsymbol{x} \tag{2}$$

Now,

$$\mathbf{v}\cdot\nabla\mathbf{v} = \mathbf{c}\times\mathbf{v} + \nabla\frac{1}{2}\mathbf{v}^2 \tag{3}$$

where $\mathbf{c} = \nabla\times\mathbf{v} = -2\mathbf{\Omega}$ denotes the vorticity. Let the velocity be decomposed by $\mathbf{v} = \mathbf{v}' - \mathbf{\Omega}\times\boldsymbol{x}$. We assume that $\mathbf{v}'$ may be obtained as the gradient of a velocity potential Φ', i.e. $\mathbf{v}' = \nabla\Phi'$. The equation of motion then gives

$$\nabla\left(\frac{\partial\Phi'}{\partial t} - \Omega\frac{\partial\Phi'}{\partial\theta} + \frac{1}{2}|\nabla\Phi'|^2\right) = \nabla\left(-\frac{p}{\rho} - gz\right) \tag{4}$$

By integration we obtain for the pressure

$$-\frac{p}{\rho}=\frac{\partial\Phi'}{\partial t}-\Omega\frac{\partial\Phi'}{\partial\theta}+\frac{1}{2}|\nabla\Phi'|^2+gz+C(t) \tag{5}$$

We next introduce $\Phi'=\phi_s+\Phi$, where $\phi_s\equiv\Omega\chi_6$ denotes the potential generated by the body when there are no waves and Φ denotes the potential due to the waves. (5) then becomes

$$-\frac{p}{\rho}=\frac{\partial\Phi}{\partial t}+\frac{\partial\phi_s}{\partial t}-\Omega\frac{\partial\Phi}{\partial\theta}-\Omega\frac{\partial\phi_s}{\partial\theta}+\frac{1}{2}|\nabla\Phi+\nabla\phi_s|^2+gz+C(t) \tag{6}$$

Both (5) and (6) are exact.

(6) was derived by Nestegård (1990) assuming a constant angular velocity.

BOUNDARY VALUE PROBLEMS

The boundary conditions for χ_6 at the body surface and the free surface read respectively,

$$\frac{\partial\chi_6}{\partial n}=\mathbf{n}\cdot(\mathbf{k}\times\boldsymbol{x})\equiv n_6 \tag{7}$$

$$\frac{\partial\chi_6}{\partial z}=0 \tag{8}$$

where in the last expression we have neglected terms being $O(\Omega^2)$.

The free surface boundary condition for Φ is obtained by applying the individual derivative, $\partial/\partial t+\mathbf{v}\cdot\nabla$, to (6) at $z=\zeta$. After linearizing we obtain

$$\frac{\partial^2\Phi}{\partial t^2}-2\Omega\frac{\partial^2\Phi}{\partial\theta\partial t}+2\nabla_h\phi_s\cdot\nabla_h\frac{\partial\Phi}{\partial t}-\frac{\partial^2\phi_s}{\partial z^2}\frac{\partial\Phi}{\partial t}+g\frac{\partial\Phi}{\partial z}=0 \quad \text{at} \quad z=0 \tag{9}$$

Here, ∇_h denotes the horizontal gradient. Let us then introduce $\Phi=\phi e^{i\omega t}$. By noting that ϕ is a function of α, Ω, $\dot{\Omega}$, ..., we obtain

$$\frac{\partial\Phi}{\partial t}=(i\omega\phi+\Omega\frac{\partial\phi}{\partial\alpha}+\dot{\Omega}\frac{\partial\phi}{\partial\Omega}+...)e^{i\omega t} \tag{10}$$

Thus, by introducing $\epsilon\equiv\omega\Omega/g$ and neglecting terms $O(\dot{\Omega})$ we obtain to leading order in ϵ

$$-K\phi+2i\epsilon\frac{\partial\phi}{\partial\alpha}-2i\epsilon\frac{\partial\phi}{\partial\theta}+2i\epsilon\nabla_h\chi_6\cdot\nabla\phi_1+i\phi\epsilon\nabla_h^2\chi_6+\frac{\partial\phi}{\partial z}=0 \quad \text{at} \quad z=0 \tag{11}$$

PERTURBATION PROCEDURE

Let $\phi=\phi^0+\epsilon\phi^1$. Then ϕ^0 satisfies

$$-K\phi^0+\frac{\partial\phi^0}{\partial z}=0 \quad \text{at} \quad z=0 \tag{12}$$

$$\frac{\partial\phi^0}{\partial n}=0 \quad \text{at the body} \tag{13}$$

ϕ^0 is composed by the incoming wave potential ϕ_I and the scattering potential ϕ_7, i.e. $\phi^0=\phi_I+\phi_7$. $K=\omega^2/g$ denotes the wave number and β the (time-dependent) wave angle in the relative frame of reference. The relative frame of reference is rotated the positive angle α relative to the fixed

frame of reference. Hence $\beta = \beta_0 - \alpha$, where β_0 denotes the wave angle in the fixed frame of reference.
The solution for ϕ^0 is obtained by means of integral equations. This integral equation shows that ϕ^0 is an implicit function of $\beta = \beta_0 - \alpha$. Thus,

$$\frac{\partial \phi^0}{\partial \alpha} = \frac{\partial \phi^0}{\partial \beta}\frac{\partial \beta}{\partial \alpha} = -\frac{\partial \phi^0}{\partial \beta} \tag{14}$$

ϕ^1 then satisfies

$$-K\phi^1 + \frac{\partial \phi^1}{\partial z} = 2i\frac{\partial \phi^0}{\partial \beta} + 2i\frac{\partial \phi^0}{\partial \theta} - 2i\nabla_h \chi_6 \cdot \nabla_h \phi^0 - i\phi^0 \nabla_h^2 \chi_6 \quad \text{at} \quad z = 0 \tag{15}$$

$$\frac{\partial \phi^1}{\partial n} = 0 \quad \text{at the body} \tag{16}$$

INTEGRAL EQUATIONS

The boundary value problem for ϕ^1 is then decomposed by introducing $\phi^1 = \phi^{11} + \phi^{12}$, where ϕ^{11} may be obtained in terms of ϕ^0. We then apply Green's theorem to ϕ^{12} and G^0, where G^0 denotes the zero speed Green function. This gives

$$\int_{S_B} \psi \frac{\partial G^0}{\partial n} dS + \int_{S_F + S(R)} \left(\psi \frac{\partial G^0}{\partial n} - G^0 \frac{\partial \psi}{\partial n}\right) dS = \left\{ \begin{array}{ll} -2\pi\psi(\boldsymbol{x}) & \boldsymbol{x} \in S_B \\ -4\pi\psi(\boldsymbol{x}) & \boldsymbol{x} \in \mathcal{V} \end{array} \right. \tag{17}$$

where S_B denotes the body surface, S_F the free surface, $S(R)$ the surface of a vertical circular cylinder with radius $R < \infty$ surrounding the body, and $\mathcal{V}$ the fluid volume. Next Green's theorem is applied to ϕ^0 and G^1, where G^1 satisfies

$$-KG^1 + \frac{\partial G^1}{\partial z} = 2i\frac{\partial G^0}{\partial \theta} \quad \text{at} \quad z = 0 \tag{18}$$

By applying the boundary conditions for ϕ^0 and G^1 at the free surface we obtain

$$\int_{S_B} \phi^0 \frac{\partial G^1}{\partial n} dS + 2i \int_{S_F} \phi^0 \frac{\partial G^0}{\partial \theta} dS + \int_{S(R)} \left(\phi^0 \frac{\partial G^1}{\partial n} - G^1 \frac{\partial \phi^0}{\partial n}\right) dS = 0 \tag{19}$$

It may then be shown that (17) and (19) gives

$$\int_{S_B} \left(\phi^1 \frac{\partial G^0}{\partial n} - \phi^0 \frac{\partial G^1}{\partial n} + 2i \frac{\partial \phi^0}{\partial \beta} \frac{\partial^2 G^0}{\partial n \partial K}\right) dS$$
$$-2i \int_{S_F} \phi^0 (\nabla_h \chi_6 \cdot \nabla_h G^0 + \frac{1}{2} G^0 \nabla_h^2 \chi_6) dS = \left\{ \begin{array}{ll} -2\pi\phi^1(\boldsymbol{x}) & \boldsymbol{x} \in S_B \\ -4\pi\phi^1(\boldsymbol{x}) & \boldsymbol{x} \in \mathcal{V} \end{array} \right. \tag{20}$$

which is the integral equation for ϕ^1.

THE DAMPING MOMENT

Consider then the vector product between the coordinate $\boldsymbol{x}$ and the equation of motion. By integrating over the fluid volume, using Gauss' theorem and the transport theorem, we obtain for the vertical component

$$M_z \equiv \mathbf{k} \cdot \int_{S_B} p(\boldsymbol{x} \times \mathbf{n}) dS = \mathbf{k} \cdot [-\rho \frac{d}{dt} \int_V \boldsymbol{x} \times \mathbf{v}' dV - \int_{S(R)} p\boldsymbol{x} \times \mathbf{n} dS - \rho \int_{S(R)} \boldsymbol{x} \times \mathbf{v}' v_n dS] \quad (21)$$

At $S(R)$ we have that $\mathbf{k} \cdot (\boldsymbol{x} \times \mathbf{n}) = 0$ and that $v_n = v_n'$. The time averaged yaw moment then becomes

$$\overline{M_z} = \rho \Omega \frac{\partial}{\partial \beta} \overline{\int_V \mathbf{k} \cdot (\boldsymbol{x} \times \mathbf{v}') dV} - \rho \int_{S(R)} \overline{v_\theta' v_n'} R dS \quad (22)$$

It may then be shown that

$$\overline{\int_V \mathbf{k} \cdot (\boldsymbol{x} \times \mathbf{v}') dV} = -\frac{\omega}{2g} \int_{S_F} [\chi_6 Im(\phi^0 \phi_{zz}^{0*}) + Im(\phi_\theta^0 \phi^{0*})] \quad (23)$$

Furthermore we have

$$\overline{v_\theta' v_n'} = \frac{1}{2R} Re[\phi_\theta^0 \phi_R^{0*}] \quad (24)$$

By then expanding the yaw-moment by

$$\overline{M_z} = \overline{M_{z0}} + \epsilon \rho B_{66} \quad (25)$$

where $\overline{M_{z0}}$ denotes the moment when the yaw motion is zero, and introducing $\phi = \phi^0 + \phi^1$, we obtain the following expression for the damping coefficient B_{66}

$$B_{66} = -\frac{1}{2} \frac{\partial}{\partial \beta} \int_{S_F} [\chi_6 Im(\phi^0 \phi_{zz}^{0*}) + Im(\phi_\theta^0 \phi^{0*})] \, dS - \frac{1}{2} \int_{S(R)} Re \, [\phi_\theta^0 \phi_R^{1*} + \phi_\theta^1 \phi_R^{0*}] \, dS \quad (26)$$

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