# TRAPPING OF WAVES BY HORIZONTAL CYLINDERS IN A CHANNEL CONTAINING TWO - LAYER FLUID

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## 1.Introduction

A horizontal channel of infinite length and depth and of constanst width contains inviscid, incompressible, two-layer fluid under gravity. The upper layer has constant finite depth and is occupied by a fluid of constant density $\rho$. The lower layer has infinite depth and is occupied by a fluid of constant density $\rho^* > \rho$. A parameter $\varepsilon = (\rho^*/\rho) - 1$ is assumed to be small. One of the fluids (upper or lower) is bounded internally by an immersed horizontal cylindrical surface $S$, which extends right across the channel and has its generators normal to the sidewalls. The free, time-harmonic oscillations of fluids having finite kinetic and potential energy (such oscillations are called trapping modes) are investigated. Trapping mode problem for a homogeneous fluid in presence of submerged cylinders or other obstacles is investigated extensively (see Evans et al. and references cited therein for bibliography). Apparently, the first treatment of this problem for the two-layer fluid is given by Kuznetsov (1993). In the case, when a cylinder is immersed in the lower fluid, it was found that under some restrictions there exist two finite sets of frequencies of trapping modes. The frequencies in the first set are close to the frequencies of trapping modes for the homogeneous fluid (when $\rho^* = \rho$). They correspond to the trapping modes of waves on the free surface of upper fluid. The frequencies in the second set are proportional to $\varepsilon$ and correspond to the trapping modes of internal waves on the interface between two fluids.

Here similar results are presented for the case, when a cylinder is immersed in the upper fluid. The general scheme of investigation is the same as in Kuznetsov (1993). First, the original problem is reduced to the problem in the layer, which contains cylinder. Then perturbation technique is applied in combination with Ursell's (1987) method of integral operators. This work was stimulated by a remark in Friis, Grue & Palm (1991), that long underwater tube bridges are proposed to be constructed across Norwegian fiords, which are often occupied by two-layer fluid (fresh-salt water).

## 2.Statement of the problem

The $xyz$-coordinates are chosen so that the $y$-axis is directed upwards and the $xz$-plane coincids with the undisturbed interface between two layers. The depth of upper layer can be assumed to be equal to one without loss of generality. Using the linear water-wave theory we consider velocity potentials of the form

$$\exp(-i\omega t)u^*(x,y)\cos kz \quad (\exp(-i\omega t)u(x,y)\cos kz)$$

for the lower (upper) fluid. Here $\omega$ is unknown radian frequency of trapping mode, and the wavenumber $k$ along the $z$-axis should be taken so that the impermeability condition holds on the sidewalls. In what follows we suppose $k$ to be prescribed, but its value is an arbitrary positive number.

The pair $\{u, u^*\}$ must be a solution of the following problem

$$u^*_{xx} + u^*_{yy} = k^2 u^* \quad \text{in } W^*, \tag{1}$$

$$u_{xx} + u_{yy} = k^2 u \quad \text{in } W, \quad \partial u/\partial n = 0 \quad \text{on } S, \quad u_y - \nu u = 0 \quad \text{when } y = 1, \tag{2}$$

$$u^*_y = u_y, \qquad \rho^*(u^*_y - \nu u^*) = \rho(u_y - \nu u) \quad \text{when } y = 0. \tag{3}$$

Here $W(W^*)$ denotes a cross-section of the region, occupied by the fluid of density $\rho(\rho^*)$, $\nu = \omega^2/g$ is the spectral parameter to be determined along with $u, u^*$ ($g$ is the acceleration of gravity). For trapped-mode solutions the motion must decay at large distances, i.e. the relations

$$u^*, \mid \nabla u^* \mid \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty, \quad \text{and} \quad u, \mid \nabla u \mid \to 0 \quad \text{as} \quad \mid x \mid \to \infty \tag{4}$$

must hold.

## 3.Perturbation method for spectral problem in the upper fluid

First, with the help of the Fourier transform one can eliminate $u^*$ from (1), (3) and (4). On this way we arrive at the following boundary condition

$$\varepsilon u_y = \nu[(1+\varepsilon)Au - u] \quad \text{when } y = 0. \tag{5}$$

Here

$$(Au)(x,0) = \frac{1}{\pi} \int_{-\infty}^{+\infty} K_0(k|x-\xi|)\, u_y(\xi, 0)\, d\xi$$

and $K_0$ is the Macdonald function. Thus, we have the boundary value problem (2), (5) with the second condition (4) at infinity.

Since the parameter $\varepsilon$ is assumed to be small, it is natural to seek eigenvalues and eigenfunctions in the form of expansions

$$\nu = \nu_0 + \varepsilon\nu_1 + \varepsilon^2\nu_2 + \dots, \qquad u = u^{(0)} + \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \dots, \tag{6}$$

which is common in the perturbation theory (see e.g. Friedrichs, 1965). Substituting (6) into (2) and (5), and equating the coefficients at the same degrees of $\varepsilon$, one obtains an infinite system of boundary value problems. The problem of the zero order is

$$u^{(0)}_{xx} + u^{(0)}_{yy} = k^2 u^{(0)} \quad \text{in } W, \quad \partial u^{(0)}/\partial n = 0 \quad \text{on } S, \tag{7}$$

$$u^{(0)}_y - \nu_0 u^{(0)} = 0 \quad \text{when} \quad y = 1, \tag{8}$$

$$\nu_0(Au^{(0)} - u^{(0)}) = 0 \quad \text{when} \quad y = 0. \tag{9}$$

The first order problem has the form

$$u_{xx}^{(1)} + u_{yy}^{(1)} = k^2 u^{(1)} \quad \text{in } W, \quad \partial u^{(1)}/\partial n = 0 \quad \text{on } S, \tag{10}$$

$$u_y^{(1)} - \nu_0 u^{(1)} = \nu_1 u^{(0)} \quad \text{when} \quad y = 1, \tag{11}$$

$$\nu_0(Au^{(1)} - u^{(1)}) = u_y^{(0)} - \nu_0 Au^{(0)} - \nu_1(Au^{(0)} - u^{(0)}) \quad \text{when} \quad y = 0. \tag{12}$$

The problem for $u^{(m)}$ $(m = 2, 3, \ldots)$ can be easily written down.

In order to fix an arbitrary factor in the expansion for $u$, which should be found from the system (7) – (9), (10) – (12) etc., it is convenient to use the linear condition $\langle u(\cdot, 1);\ u^{(0)}(\cdot, 1)\rangle = 1$, where $\langle \cdot\ ;\ \cdot\rangle$ is the scalar product in $L_2(+\infty, -\infty)$. The last equality combined with the following normalization condition $\langle u^{(0)}(\cdot, 1);\ u^{(0)}(\cdot, 1)\rangle = 1$, gives

$$\langle u^{(1)}(\cdot, 1)\ ;\ u^{(0)}(\cdot, 1)\rangle = \langle u^{(2)}(\cdot, 1)\ ;\ u^{(0)}(\cdot, 1)\rangle = \ldots = 0. \tag{13}$$

The problem (7) – (9) has a finite set of positive point eigenvalues $\{\nu_0^{(+)}\}$, because this problem is another form of the problem on trapping modes above the cylinder immersed in the homogeneous fluid $(\rho^* = \rho)$. If $\nu_0^{(+)}$ is a positive non-degenerate eigenvalue for (7) – (9) and $u_+^{(0)}$ is the corresponding eigenfunction, then (12) takes the form

$$Au_+^{(1)} - u_+^{(1)} = -u_+^{(0)} + [\nu_0^{(+)}]^{-1}\partial u_+^{(0)}/\partial y \quad \text{when} \quad y = 0. \tag{14}$$

The problem (10), (11), (14) is solvable under the orthogonality condition

$$\nu_1^{(+)} - \langle u_+^{(0)}(\cdot, 0);\ u_+^{(0)}(\cdot, 0)\rangle + [\nu_0^{(+)}]^{-1}\langle u_+^{(0)}(\cdot, 0);\ (\partial u_+^{(0)}/\partial y)(\cdot, 0)\rangle = 0,$$

which expresses $\nu_1^{(+)}$. Then $u_+^{(1)}$, can be found uniquely in view of (13). Thus, the terms in the expansions (6) can be successively determined, what gives $\nu^{(+)} > 0$ and $u_+$ to any necessary accuracy when $\varepsilon$ is small enough.

If $\nu_0^{(0)} = 0$, then (9) trivially holds and (12) takes the form

$$u_y^{(0)} = \nu_1^{(0)}(Au^{(0)} - u^{(0)}) \quad \text{when} \quad y = 0. \tag{15}$$

This boundary condition, complemented by (7) and by the homogeneous Neumann condition on $y = 1$ (it follows from (8)), forms a spectral problem. It differs by the term $Au^{(0)}$ in (15) from the problem on trapped modes above a cylinder immersed in a homogeneous fluid of finite depth.

## 4.The spectral problem for $\nu_1^{(0)}$

Following Ursell (1987) we seek $u_0^{(0)}$ in the form of a single layer Green potential

$$u_0^{(0)}(x, y) = (V\mu)(x, y) = 1/\pi \int_{-\infty}^{+\infty} \mu(\xi) g(x, y; \xi, 0)\, d\xi,$$

where $\mu \in L_2(-\infty, +\infty)$ and $g(\dots)$ is Green's function satisfying (10) and the homogeneous Neumann condition on $y = 1$ and on $y = 0$ (with exception for $x = \xi$ in the last case). This Green function is constructed by Ursell (1987). Since $\partial V\mu/\partial y = -\mu$ when $y = 0$, then (15) yeilds

$$\mu = \nu_1^{(0)} T\mu = \frac{\nu_1^{(0)}}{\pi} \int_{-\infty}^{+\infty} [K_0(k|x-\xi|) + g(x, 0; \xi, 0)]\, \mu(\xi)\, d\xi. \tag{16}$$

In the same way as in Kuznetsov (1993) one can show, that$\pi T$differs from the operator $2G$ with the kernel $2g(x, 0; \xi, 0)$ by an operator, whose norm exponentially decays as $k \to \infty$. On the other hand, Ursell (1987) proved that $G$ has a finite set of positive point eigenvalues. Hence, there is a finite set $\{\nu_1^{(0)}\}$ of positive eigenvalues for $T$, when $k$ is large. Applying the same procedure as in § 3, we arrive at the eigenvalue expansion $\nu^{(0)} = \varepsilon\nu_1^{(0)} + \varepsilon^2\nu_2^{(0)} + \dots$, which is positive for sufficiently small $\varepsilon$. Then, $\omega_i = (g\nu^{(0)})^{1/2}$ is the frequency of trapped mode of internal waves on the interface.

It is easy to see that $\omega_i/\omega_s \approx (\varepsilon/2)^{1/2}$, when $\varepsilon$ is small enough and $k$ is large enough. Here $\omega_s$ is a trapping mode frequency for waves on the free surface of the following finite depth channel. We have to topsyturvy the upper fluid layer with the cylinder and to supply it with the rigid horizontal bottom.

## 5.Conclusion and discussion

Kuznetsov (1993) demonstrated that there exist trapping modes of internal waves when a cylinder is immersed in the lower infinite depth layer. Here the same is shown to be true when the upper layer contains a cylinder. For both positions of cylinder the relation $\omega_i/\omega_s \approx (\varepsilon/2)^{1/2}$ is valid, but with different meaning for $\omega_s$. It should be reminded that in Kuznetsov (1993) $\omega_s$ denotes the trapping mode frequency for waves on surface of the lower fluid in absence of the upper layer. In § 4 the meaning of $\omega_s$ is quite unlike to the cited above.

The existence of trapping modes of both considered types can be demonstrated similarly for any above mentioned cylinder's position in a two-layer fluid of finite depth. The method developed here can be also applied for finding trapping mode frequencies of internal waves in the case, when a cylinder intersects the free surface of the upper layer. It is interesting to note, that there are no trapping modes of surface waves, if the latter configuration satisfies John's condition. This follows from a result proved by McIver (1991, Appendix A) on absence of trapping modes in the homogeneous fluid in presence of such surface-piercing cylinder.

## References


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Ursell, F. 1987 *J.Fluid Mech.* **183**, 421–437.

## DISCUSSION

**Palm E.**: In your proof $\varepsilon$ is assumed small. Do you have any idea about how large $\varepsilon$ may be for your proof to be true?

**Kuznetsov N.**: For practically interesting case of fresh-salt water we have $\varepsilon$ in the interval from 0.2 to 0.4. These values are small enough for convergence of perturbation expansion as it is known from other examples.

**Evans D.V.**: Does your proof hold for arbitrary (small) $\varepsilon$ or do you have to assume that the expansion in $\varepsilon$ is convergent?

**Kuznetsov N.**: If a cylinder is immersed in the lower fluid, then the result can be easily transformed to become a rigorous mathematical theorem. Only a formal asymptotic expansion is obtained for the second case, when the upper fluid contains cylinder.