A relation between the three-dimensional and the two-dimensional Green functions of the Neumann-Kelvin Problem

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1 Introduction

We are concerned with the linearized wave resistance problem (also called Neumann-Kelvin problem), i.e., the perturbation of a uniform flow by a fixed rigid body assumed immersed or not. This problem calls for a preliminary study of the associated Green function, that is the solution of the problem of an immersed source in the same flow. It has already proved to be of great use for the two-dimensional problem.

Indeed, in the two-dimensional case, the Green function is regular except near the source. It can be used to find an integral representation formula (of the two-dimensional Neumann-Kelvin solution) that enables us to set the problem in a bounded domain. By this method, the solution can be theoretically studied and numerically computed.

In the three-dimensional case, the study is not so easily performed. The Green function is well-known (see e.g.[1]). Some of its properties have been studied (see e.g.[2]). It has been shown that it is not regular, so that all the results deduced in the two-dimensional case, have not been obtained by our method.

In this paper, we state that there is a relation between the Green functions of the three-dimensional and the two-dimensional problems: they are related through the Radon transform.

2 Application of the Radon transform to the Green function of the three-dimensional problem

2.1 The Green functions

We first describe the 3D case. (x,y) are the horizontal coordinates of point P and z is the vertical one. The perfect fluid domain is located in the half-space $\{z \leq 0\}$. Its boundary is located at $z=0$. The flow is parallel to the x-axis and its velocity is V_0. We set $\nu = \frac{g}{V_0^2}$. The Green function $G_\nu^3(M,P)$ satisfies problem $(\mathcal{P}_\nu^3)$:

$$(\mathcal{P}_\nu^3)\begin{cases} \text{(a)} & \Delta_P G_\nu^3(M,P) = \delta_M(P) & \text{in } z<0, \\ \text{(b)} & (\partial_x^2+\nu\,\partial_z)G_\nu^3(M,P) = 0 & \text{on } z=0, \\ \text{(c)} & \lim\limits_{z\to-\infty} \partial_z G_\nu^3(M,P) = 0, & \\ \text{(d)} & \nabla_P G_\nu^3(M,P) = O(\frac{1}{\sqrt{x^2+y^2}}) & x\to+\infty, \forall(y,z), \end{cases}$$

where $\delta_M(P)$ denotes the Dirac measure at point M. Since the problem is invariant through any horizontal translation, we can reduce the study to the case of a source located at point $M=(0,0,z_M)$.

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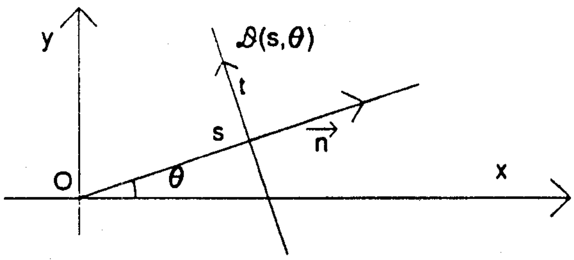
In the 2D case, (s,z) are the coordinates of point P. The Green function $G^2_\nu(M,P)$ satisfies problem $(\mathcal{P}^2_\nu)$:

$$(\mathcal{P}^2_\nu)\begin{cases} \text{(a)} & \Delta_P G^2_\nu(M,P) = \delta_M(P) & \text{in } z<0, \\ \text{(b)} & (\partial^2_s + \nu\,\partial_z) G^2_\nu(M,P) = 0 & \text{on } z=0, \\ \text{(c)} & \lim_{z\to-\infty} \partial_z G^2_\nu(M,P) = 0, & \\ \text{(d)} & \lim_{s\to+\infty} \nabla_P G^2_\nu(M,P) = 0 & \text{in } z<0. \end{cases}$$

$G^2_\nu(M,P)$ is the calm upstream solution. If we replace $\lim_{s\to+\infty}$ by $\lim_{s\to-\infty}$ in equation (d), $G^2_\nu(M,P)$ is the calm downstream solution. From now on, we will denote problem $\mathcal{P}^2_{\frac{\nu}{\cos^2\theta}}$ by $\mathcal{P}^2_{\nu,\theta}$. We now introduce the Radon transform.

2.2 The Radon transform

For all $f(x,y)$, $\mathcal{R}f$ is the integral of f over $\mathcal{D}(s,\theta)$. $\mathcal{D}(s,\theta)$ is the straight line parallel to and positively oriented by the vector $(-s\sin\theta, s\cos\theta)$ and whose distance from the origin is $|\ s\ |$.



Definition 1

(1) $$(\mathcal{R}_{x,y}f)(s,\theta) = \int_{-\infty}^{+\infty} f(s\cos\theta - t\sin\theta, s\sin\theta + t\cos\theta)dt.$$

2.3 Relation between the Green functions

Theorem 1

$$(\mathcal{R}_{x,y}G^3_\nu(x,y,z,z_M))(s,\theta) = G^2_{\nu,\theta}(s,z,z_M),$$

where $G^2_{\nu,\theta}$ is the calm upstream solution of $\mathcal{P}^2_{\nu,\theta}$ when $\theta\in]-\frac{\pi}{2},\frac{\pi}{2}[$ and the calm downstream one when $\theta\in]\frac{\pi}{2},\frac{3\pi}{2}[$.

Let us notice that $\frac{\nu}{\cos^2\theta}$ corresponds to a two-dimensional flow whose velocity is $V_0\cos\theta$, that is the projection of the three-dimensional flow onto the plane normal to the vector $(-\sin\theta,\cos\theta)$.

3 Sketch of the proof of the theorem

3.1 Properties of the Radon transform

Let us recall some basic properties of the Radon transform. All the details can be found in [3]. There is a relation between the Radon transform and the Fourier transform; if we denote the n-dimensional Fourier transform by the following formula,

$$\forall v\in\mathbb{R}^n, (\mathcal{F}_u f)(v) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-i(u|v)} f(u)du,$$

we obtain:

(2) $$\forall(\rho,\theta)\in\mathbb{R}\times[0,2\pi],\ (\mathcal{F}_{x,y}f)(\rho\cos\theta,\rho\sin\theta) = \frac{1}{\sqrt{2\pi}}(\mathcal{F}_s(\mathcal{R}_{x,y}f)(s,\theta))(\rho).$$

This formula (2) can allow us to prove that the Radon transform is 1 to 1.

We now gather some formulas we will use to prove the theorem. Using the definition of the Radon transform (1), we obtain:

$$\text{(3)} \qquad \mathcal{R}_{x,y}(\partial_x f)(s,\theta) = \cos\theta\,\partial_s(\mathcal{R}_{x,y}f)(s,\theta) \text{ and } \mathcal{R}_{x,y}(\partial_y f)(s,\theta) = \sin\theta\,\partial_s(\mathcal{R}_{x,y}f)(s,\theta).$$

Reiterating, we have:

$$\text{(4)} \qquad \mathcal{R}_{x,y}(\Delta_{x,y} f)(s,\theta) = \partial_s^2(\mathcal{R}_{x,y}f)(s,\theta) \text{ and } \mathcal{R}_{x,y}(\partial_x^2 f)(s,\theta) = \cos^2\theta\,\partial_s^2(\mathcal{R}_{x,y}f)(s,\theta).$$

We also obtain:

$$\text{(5)} \qquad \forall q \in \mathbb{N},\ \mathcal{R}_{x,y}(\partial_z^q f)(s,\theta) = \partial_z^q(\mathcal{R}_{x,y}f)(s,\theta),$$

$$\text{(6)} \qquad \mathcal{R}_{x,y}(\delta(x=0, y=0, z=z_M))(s,\theta) = \delta(s=0, z=z_M).$$

3.2 Application

Here, we state that we can transform the three-dimensional problem into the two-dimensional one. In a first step, we try to apply the Radon transform to the equations of $\mathcal{P}_\nu^3$. But we don't see how the upstream condition, i.e. equation (d), can be easily transformed. This leads us to only consider the first three equations of the three-dimensional problem which are however easily transformed into the first three equations of the two-dimensional problem. In a second step, we verify that the last equation (d) of the two-dimensional problem is also satisfied when the calm upstream condition (d) of $\mathcal{P}_\nu^3$ is satisfied.

We consider problem $(\tilde{\mathcal{P}}_\nu^3)$, which consists in the same equations as $\mathcal{P}_\nu^3$ save equation (d). $\tilde{G}_\nu^3(M,P)$ is a solution of $(\tilde{\mathcal{P}}_\nu^3)$:

$$(\tilde{\mathcal{P}}_\nu^3)\ \{(\mathrm{a}),(\mathrm{b}),(\mathrm{c}).$$

If we take the Radon transform of $\tilde{G}_\nu^3(M,P)$ with respect to (x,y), we obtain, thanks to formulas (4),(5),(6), $\forall\theta \in [0,2\pi], \mathcal{R}_{x,y}(\tilde{G}_\nu^3(M,P))(s,\theta)$ is solution of $\tilde{\mathcal{P}}_{\nu,\theta}^2$. $\tilde{\mathcal{P}}_{\nu,\theta}^2$ is the following problem:

$$(\tilde{\mathcal{P}}_{\nu,\theta}^2)\begin{cases} (\mathrm{a}) & \Delta_P \tilde{G}_{\nu,\theta}^2(M,P) = \delta_M(P) \quad \text{in } (z<0) \\ (\mathrm{b}) & (\cos^2\theta\,\partial_s^2 + \nu\,\partial_z)\tilde{G}_{\nu,\theta}^2(M,P) = 0 \quad \text{on } (z=0) \\ (\mathrm{c}) & \lim\limits_{z\to-\infty} \partial_z \tilde{G}_{\nu,\theta}^2(M,P) = 0. \end{cases}$$

We recall that (s,z) are the coordinates of point P. Problem $\tilde{\mathcal{P}}_{\nu,\theta}^2$ consists in equations of $\mathcal{P}_{\nu,\theta}^2$ save equation (d). We know its solutions that we denote by $\tilde{G}_{\nu,\theta}^2(M,P)$. We have obtained:

Lemma 1

$$\mathcal{R}_{x,y}(\tilde{G}_\nu^3(x,y,z,z_M))(s,\theta) = \tilde{G}_{\nu,\theta}^2(s,z,z_M).$$

The Fourier transform with respect to s of $\tilde{G}_{\nu,\theta}^2$ is:

$$\text{(7)} \qquad \begin{aligned} \mathcal{F}_s\left(\tilde{G}_{\nu,\theta}^2(s,z,z_M)\right)(\xi) &= -\frac{1}{\sqrt{2\pi}} P.V.\left(\frac{e^{|\xi|(z+z_M)}}{2|\xi|} + \frac{e^{-|\xi||z-z_M|}}{2|\xi|}\right) \\ &+ \frac{1}{\sqrt{2\pi}} \lim_{\epsilon\to0+}\left(\frac{e^{|\xi|(z+z_M)}}{|\xi| - \frac{\nu+i\epsilon}{\cos^2\theta}}\right) \\ &+ \mathcal{F}_s\left(\sqrt{2\pi}(\alpha(\theta)-\alpha(\theta+\pi))e^{\frac{\nu(z+is)}{\cos^2\theta}}\right)(\xi). \end{aligned}$$

By using (2), (7) and Lemma 1, we obtain $\mathcal{F}_{x,y}\left(\tilde{G}_\nu^3\right)$. By 2-dimensional inverse Fourier transform, this formula gives an expression of $\tilde{G}_\nu^3$, determined provided the function $\alpha(\theta)$ is fixed $\forall\theta \in [0,2\pi] - \{-\frac{\pi}{2},\frac{\pi}{2}\}$.

In a second step, we determine this function $\alpha(\theta)$ so that $\tilde{G}_\nu^3$ satisfies equation (d) of problem $\mathcal{P}_\nu^3$. In fact, we only have to study the asymptotic behaviour of $\tilde{G}_\nu^3$, by means of the stationary phase method. It follows that $\alpha(\theta) = -i\sqrt{\frac{\pi}{2}}e^{\frac{\nu z_M}{\cos^2\theta}}$ for $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ and $\alpha(\theta) = i\sqrt{\frac{\pi}{2}}e^{\frac{\nu z_M}{\cos^2\theta}}$ for $\theta \in]\frac{\pi}{2}, \frac{3\pi}{2}[$.We find that the Green function $G^3_\nu(M,P)$, solution of $\mathcal{P}^3_\nu$ is:

$$(8)\begin{cases} G^3_\nu(x,y,z,z_M) = \frac{1}{4\pi}\left(\frac{1}{\sqrt{x^2+y^2+(z+z_M)^2}} - \frac{1}{\sqrt{x^2+y^2+(z-z_M)^2}}\right) \\ \quad + \frac{\nu}{4\pi^2}\lim_{\epsilon\to 0^+}\int_{-\infty}^{+\infty}\int_0^\infty \left(\frac{e^{\rho(z+z_M)(1+t^2)}}{\rho-(\nu+i\epsilon)}\right)\left(e^{i\rho\sqrt{1+t^2}(x+ty)} + e^{-i\rho\sqrt{1+t^2}(x+ty)}\right) d\rho dt \\ \quad - \frac{i\nu}{2\pi}\int_{-\infty}^{\infty} e^{\nu\left((z+z_M)(1+t^2)+i\sqrt{1+t^2}(x+ty)\right)} dt. \end{cases}$$

Besides, $\mathcal{R}_{x,y}(G^3_\nu(x,y,z,z_M))(s,\theta)$ also satisfies equation (d) of $\mathcal{P}^2_{\nu,\theta}$, that is the calm upstream condition or the calm downstream one. Indeed, we verify that $\alpha(\theta)$ is the same as the one directly given by solving problem $\mathcal{P}^2_{\nu,\theta}$. We obtain the theorem given in section 2, i.e.:

$$(\mathcal{R}_{x,y}G^3_\nu(x,y,z,z_M))(s,\theta) = G^2_{\nu,\theta}(s,z,z_M).$$

Let us notice that this result shows the equations (d) of $\mathcal{P}^3_\nu$ and $\mathcal{P}^2_{\nu,\theta}$ are related through the Radon transform, which is not obvious by a direct computation.

Conclusion

Our final purpose is actually to study how the wave resistance depends on the flow velocity, and to determine the local extrema of this function. Our approach is to extend the problem to complex values of the flow velocity. Each maximum then appears as the trace of a singularity of this extended problem. A value of the velocity for which a singularity occurs is called a resonance.

In the two-dimensional Neumann-Kelvin problem, the existence of resonances has been obtained (see e.g. [[4]]). The method employed is based on a property of G^2_ν. Equation (7) gives that G^2_ν consists in a Green function of scattered type (i.e. a function that behaves as outgoing plane waves at infinity) plus a plane wave. It is shown in [[4]] that such a decomposition enables one to decompose the solution of the Neumann-Kelvin problem into a solution of scattered type and a plane wave. This decomposition leads to the existence of the resonances.

Consider now equation (8). It shows that the same kind of decomposition holds. Although the scattered type function and the plane waves consist in a sum depending upon $t = \tan\theta$, we hope to obtain the same decomposition for the solution of the three-dimensional Neumann-Kelvin problem and finally the existence of resonances.

References

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DISCUSSION

Yue. D.K.P.: Could you comment on the practical diffculty you may have near $\theta = \pm\frac{\pi}{2}$? It should be noted that physically nothing special happens at $\theta = \pm\frac{\pi}{2}$.

Doutreleau Y.: The point is that for $\theta = \pm\frac{\pi}{2}$, the free surface condition in the 2D problem degenerates, because there is no more 2nd order derivative with respect to the horizontal coordinate in it. So the Green function obtained is completely different and it must be equal to the Rankine source. However, I'm not sure that when θ goes to $\theta = \pm\frac{\pi}{2}$, the 2D Green function might tend to the Rankine source in a distributional way.

Newman N.: Is it possible to find new algorithms more suitable for computations using these transforms?

Doutreleau Y.: I can't be either affirmative in negative, because I didn't investigate this question. But what is sure is that the invese Radon Transform that needs to make two Founier transforms and then to evaluate on integral must be complicate. Notice that it would be very surprising if it was easy to do it.