COMPUTATION OF THE FINITE DEPH TIME-DOMAIN GREEN FUNCTION IN THE SMALL TIME RANGE

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The first formulation of a time-domain Green function of unsteady free-surface hydrodynamics seems to appear in Brard (1948) in the case of infinite water depth. Finkelstein, in 1957, gave the expressions for both the finite and infinite water depth, in two and three dimensions. Since that time, a lot of alternative formulations have been developed in order to improve the efficiency of the computation of this function in the infinite water depth case. (see e.g:Newman (1985), Liapis (1986), King (1987)). Further methods were proposed to avoid *in-line* calculations of this function in time-domain seakeeping computations: the tabulation of the function (Ferrant 1988) and the identification of the function (Clément 1991-1992). Thus, owing to these studies, the time-domain modelization of the seekeeping problem has became a valuable alternative to the traditional frequency domain approach, at least when the water depth may be considered to be infinite.

When this hypothesis does not hold, one must use the finite depth time-domain Green function instead of the previous one, and this function is far more difficult to evaluate numerically. Only a few results have been published about practical methods and algorithms for its computation (Newman 1990). So we began last year to study this problem , and we present here the very first results we have obtained for the small time range which is evidently the easiest part of the whole job.

THE MEMORY PART OF THE GREEN FUNCTION.

We are concerned here with the potential generated in $M(X,Y,Z)$ at time T by a source of unit impulsive strength $\delta(0)$ located at $M'(X',Y',Z')$. This Green function is the sum of an impulsive term, which will not be investigated here due to its relative simplicity, and a memory part given by:

$$\tilde{G}_h(M,M',T)=\int_0^{\infty}\sqrt{K}\,\frac{\sin\left(T\sqrt{K\tanh K}\right)}{\cosh^2 K\ \sqrt{\tanh K}}\cosh K(Y+1)\cosh K(Y'+1)\,J_0(KR)\,dK \qquad (1)$$

$R=\sqrt{(X-X')^2+(Z-Z')^2}$ is the horizontal distance between the points, and T is the time since the impulsive birth of the unit strength source.

Distances are nondimensionalized with respect to the depth h; thus we have: $-1\leq Y\leq 0$, and $-1\leq Y'\leq 0$. Let us recall here the expression of the infinite depth time domain Green function:

$$\tilde{G}_{\infty}(M,M',T)=\int_0^{\infty}\sqrt{K}\sin\left(T\sqrt{K}\right)\exp[K(Y+Y')]\,J_0(KR)\,dK \qquad (2)$$

the evaluation of which will be performed by a bilinear interpolation process in a precomputed table, (see Ferrant 1988) . Given the source point $M'(X',Y',Z')$ and the field point $M(X,Y,Z)$ we define the following points:

$$\begin{cases} M_1'(X',-Y',Z') & \text{image of the source point about the free surface } Y=0 \\ M_2'(X',-2-Y',Z') & \text{image of the source point about the bottom } Y=-1 \\ M_3'(X',-4-Y',Z') & \text{image of the source point about the image of the free surface } Y=-2 \end{cases}$$

$$\begin{cases} M_1(X,Y,Z) & \text{image of the field point about the free surface } Y=0 \\ M_2(X,-2-Y,Z) & \text{image of the field point about the bottom } Y=-1 \end{cases}$$

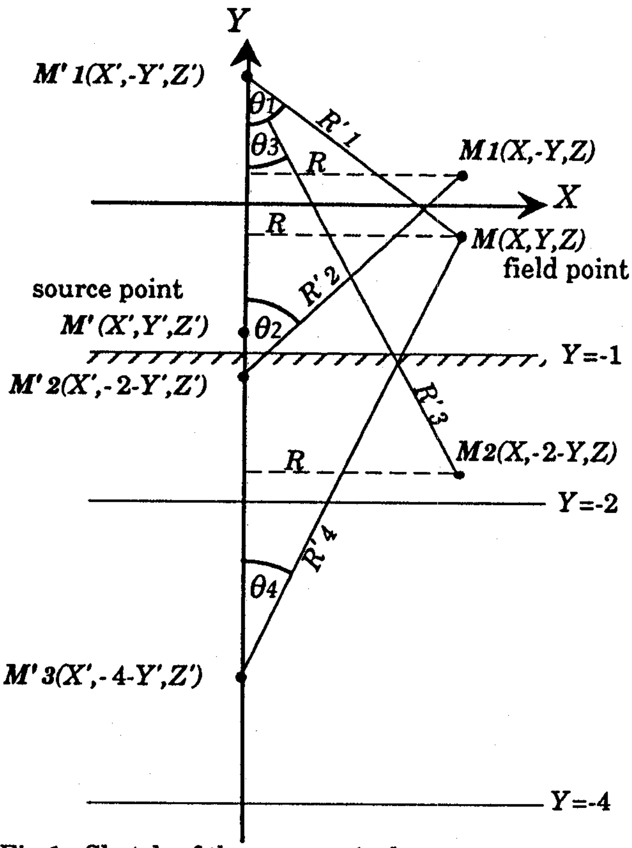

Fig.1 : Sketch of the geometrical parameters

The four angles $\theta_1, \theta_2, \theta_3, \theta_4$ are defined as in fig.1. We introduce four new vertical variables Y_k

$$\begin{cases} Y_1 = Y + Y' & -2 \le Y_1 \le 0 \\ Y_2 = -(Y' - Y + 2) & -3 \le Y_2 \le -1 \\ Y_3 = -(Y - Y' + 2) & -3 \le Y_3 \le -1 \\ Y_4 = -(Y + Y' + 4) & -4 \le Y_3 \le -2 \end{cases}$$

By first expanding the hyperbolic cosines , $\tilde{G}_h$ is expressed as the sum of four terms:

$$\tilde{G}_h(M,M',T) = \sum_{k=1}^{k=4} {}_k\tilde{G}_h \qquad \text{with}$$

$${}_k\tilde{G}_h = \int_0^\infty \sqrt{K}\, \frac{\sin\left(T\sqrt{K\tanh K}\right)}{\sqrt{\tanh K}} \frac{\exp(KY_k)}{(1+\exp(-2K))^2} J_0(KR)dK$$

Proceeding the same way than for the infinite depth function, the sine function is expanded in series of powers of the time variable T:

$${}_k\tilde{G}_h = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{T^{(2i-1)}}{(2i-1)!} \int_0^\infty K^i \exp(KY_k) \frac{(1-\exp(-2K))^{i-1}}{(1+\exp(-2K))^{i+1}} J_0(KR)dK$$

Expanding now the function $\frac{(1-\exp(-2K))^{i-1}}{(1+\exp(-2K))^{i+1}}$ in series of powers of the variable exp(-2K) we obtain:

$$\frac{(1-\exp(-2K))^{i-1}}{(1+\exp(-2K))^{i+1}} = \sum_{j=1}^{\infty} a_{ij} \exp(-2jK)$$

where the coefficients matrix a_{ij} is easilly built by the recurrence relationship:

$$a_{ij} = (-1)^{j+1} \left\{ \left|a_{(i-1)j}\right| + 2 \sum_{p=1}^{p=j-1} \left|a_{(i-1)p}\right| \right\} \qquad \text{with:} \quad a_{i1} = 1 \; ; \; a_{1j} = (-1)^{j+1}\left(1 + \left|a_{1(j-1)}\right|\right) \quad (3)$$

Thus, finally, the Green function is given by the following expansion:

$$\tilde{G}_h = \sum_{k=1}^{k=4} \sum_{i=1}^{\infty} (-1)^{i+1} \frac{T^{(2i-1)}}{(2i-1)!} \sum_{j=1}^{\infty} a_{ij} \int_0^\infty K^i \exp[-K(2j-2-Y_k)] J_0(KR)dK \qquad (4)$$

The next stage is the evaluation of the integrals in the above expression. We propose the following two methods.

LEGENDRE POLYNOMIALS. This first method is based on the identity (Gradshteyn & Ryzhik 6.624):

$$\int_0^{\infty} \exp(-\mu k) J_0(k\sqrt{1-\mu^2}) k^{\nu} dk = \Gamma(\nu+1) P_{\nu}(\mu) \tag{5}$$

Plugging(5) in (4) leads, after some manipulations and reordering, to the result:

$$\begin{aligned}\tilde{G}_h(M,M',T) &= \tilde{G}_{\infty}(M,M',T) + \tilde{G}_{\infty}(M,M_2',T) + \tilde{G}_{\infty}(M_2,M',T) + \tilde{G}_{\infty}(M_1,M_3',T) + \\ &+ \sum_{k=1}^{k=4} \sum_{i=1}^{\infty} (-1)^{i+1} \frac{i!}{(2i-1)!} T^{2i-1} \sum_{j=2}^{\infty} a_{ij} \frac{P_i(\mu_{jk})}{\rho_{jk}^{\,i+1}}\end{aligned} \tag{6}$$

with $y_{jk} = [2(j-1) - Y_k]$, $\rho_{jk} = \sqrt{(X-X')^2 + y_{jk}^2 + (Z-Z')^2}$ and $\mu_{jk} = y_{jk}/\rho_{jk}$

The four infinite depth Green function terms represent the contribution of j=1 in the summations. They can be dropped out if one starts the last sum at j=1 instead of j=2.

SUCCESSIVE DERIVATIVES. In this second approach we use the following identity (Gradshteyn & Ryzhik 6.621):

$$\int_0^{\infty} k^m \exp(-\alpha k) J_{\nu}(\beta k) dk = (-1)^m \beta^{-\nu} \frac{d^m}{d\alpha^m}\left[\left(\sqrt{\alpha^2+\beta^2}-\alpha\right)^{\nu}\left(\alpha^2+\beta^2\right)^{-1/2}\right] \tag{7}$$

which lead to the formula:

$$\tilde{G}_h(M,M',T) = -\sum_{k=1}^{k=4} \sum_{i=1}^{\infty} \frac{T^{2i-1}}{(2i-1)!} \sum_{j=1}^{\infty} a_{ij} \frac{d^i}{d y_{jk}^i}\left(\frac{1}{\rho_{jk}}\right) \tag{8}$$

The successive derivatives in the last summation may be computed by:

$$\frac{d^n}{d\alpha^n}\left(\alpha^2+R^2\right)^{-\frac{1}{2}} = \sum_{m=0}^{m=\mathrm{Int}(\frac{n}{2})} b_{nm} \frac{\alpha^{n-2m}}{\left(\alpha^2+R^2\right)^{\frac{2n-2m+1}{2}}} \tag{9}$$

where the constant coefficients b_{nm} derive from a simple recurrence relation.

NUMERICAL RESULTS.

Expressions (6) and (8)-(9) above give the Green function as a series in positive odd powers of the time variable T. The infinite series in i and j are troncated at $i=I$, respec. $j=J$. In both methods, recurrence relations are used to calculate the successive terms of the inner series. So, it is very easy to implement an algorithm where I and J may be increased as far as necessary to obtain a convergence. The coefficient matrices a_{ij} and b_{nm} are constant. Thus they were tabulated once for all and stored in a permanent file.

Both methods have been implemented and tested for various combinations of source submergence, radial distance and water depth. In each case, we first computed a reference solution by inverse Fourier transforming the corresponding frequency domain Green function for which robust algorithms were developped in our laboratory (Delhommeau 1989).

Then the computations were performed using successively the *Legendre polynomials* (6) and the *successive derivatives* (8)-(9) methods to evaluate the coefficients of the series in T. The time variable T was increased by constant steps from zero up to a value were the series began to diverge.

We give herein the results for two typical configuration: one with a relatively large value of the horizontal distance: R=5, the other with a small value: R=0.25. For the first one (Fig.2) we see that the

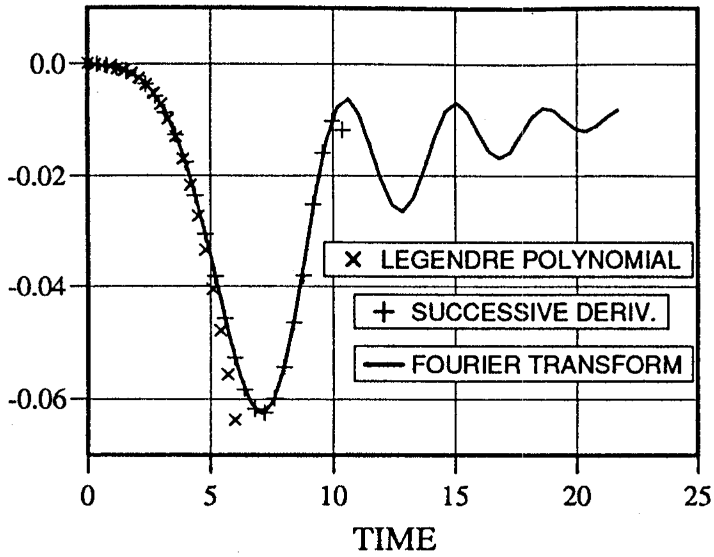

Fig.2: The time-domain Green function for: $M'(0,-1,0)$ and $M(10,-1,0)$, with $h=2$

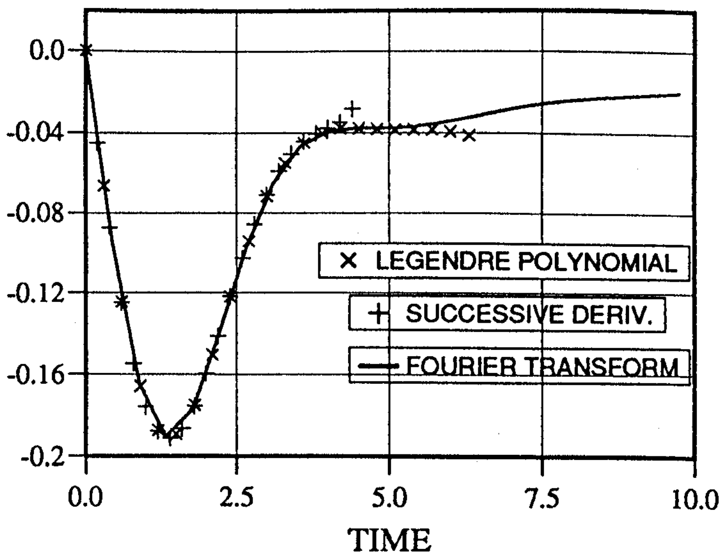

Fig.3: The time-domain Green function for: $M'(0,-1,0)$ and $M(0.5,-1,0)$, with $h=2$

successive derivatives method give accurate results up to T=10, when the other method begins to fail to converge before T=4. For the moderate values of R (fig.3), the reverse situation is observed: The Legendre polynomials method allows to reach greater time values than the derivatives. Then we suggest to keep both the algorithms in a general routine, and to switch between them regarding the value of the horizontal distance R.

In term of computing time the Legendre polynomial method was found to run approximately two times faster than the derivatives. At the moment, without any optimization, the cpu ratio with the infinite depth time-domain Green function lies between twenty and fourty for a five digit accuracy in both cases.

We are aware that the small time range was the simplest angle of attack of the problem. So, after this warm-up, we have begun to study the moderate and large time ranges. We hope to be able to give useful innovative results at the next Workshop.

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DISCUSSION

Korobkin A.: Is there overlapping time-interval of your expansion and that presented by Prof. Newman?

Clement A.: We surely need an expansion for large time, overlapping with the short time expansion presented here. So, we shall use expansions, like those presented by Newman, which effectively overlap, for large time calculations. I want to point out that our formulation works from small to "medium" time range. If you look at the results we give, you will see that we reach a value of $R/T = 0.5$ in the large radius test case and $R/T = 0.05$ in the other case, passing through the "wave front" value $R/T = 1$ without any numerical trouble. So the need for an intermediate expansion is not so evident.

Yue. D.K.P.: Your numerical results are for relatively deep submergence of the field and source points (as a result the Green function near $x/t \sim 1$ does not resemble a wave front at all) – do your algorithms have similarly numerical performance for the case of small $Y + Y'$?

Clement A.: We did not check the performance of an algorithm for all the possible combinations of the parameters $R, Y+Y', T \cdots$. So, I cannot answer precisely your question about the convergence behavior in that case. But, regarding what happens in the infinite depth case, we can anticipate a more and more oscillating Green function as both points approach the free surface, and probably a more difficult convergence of the waves. We shall investigate this point precisely in a near future to be able to give a better answer to your question.