Trapped Modes Above A Submerged Flat Plate

N.F. Parsons *and* P.A.Martin
Department of Mathematics, University of Manchester, Manchester M13 9PL, England

1 Introduction

We consider the trapping of surface water waves above a thin plate, Γ. In a paper by Linton & Evans [2], strong numerical evidence is given for the existence of trapped modes above a submerged, horizontal, flat plate, in finite depth water. The method uses matched eigenfunctions. Here, we treat the case of deep water, and allow the plate some inclination to the horizontal. The method should also be able to treat curved plates, although we do not consider them here. The method that we use involves the reduction of the problem to the solution of a homogeneous, hypersingular integral equation for the discontinuity in potential across the plate. The trapped–mode frequencies are then given by the existence of a non-trivial solution to the integral equation.

As with some work carried out on a related scattering problem, see Parsons & Martin [3] and [4], we approximate the solution by means of a truncated series of Chebyshev polynomials of the second kind, multiplied by a suitable weight. Collocation then gives a homogeneous matrix equation for the unknown coefficients. A non-trivial solution is then given by the vanishing of the determinant.

2 Formulation

We choose the mean free surface to be the (x, z)-plane, and take y vertically down into the fluid. We suppose that the two edges of the plate are parallel to the z-axis. Assuming classical, linear water wave theory, and time-harmonic motion with angular frequency ω, we write

$$\Phi(x, y, z, t) = \Re\left\{\phi(x, y)e^{i(lz-\omega t)}\right\},$$

where l is the wavenumber in the z–direction. It is known that trapped modes are not possible for $K(=\omega^2/g) > l$, where g is the acceleration due to gravity. If we therefore let $K = l\cos\beta$, then we see that $\phi(x, y)$ must satisfy

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} - l^2\phi = 0 \qquad \text{in the fluid,} \tag{1}$$

$$l\cos\beta\,\phi + \frac{\partial\phi}{\partial y} = 0 \qquad \text{on the free surface,}\quad y = 0 \tag{2}$$

and

$$\frac{\partial\phi}{\partial n} = 0 \qquad \text{on the plate,}\quad \Gamma; \tag{3}$$

$\partial/\partial n$ is normal differentiation on Γ. For trapped modes, we also require

$$\phi, |\nabla\phi| \to 0 \qquad \text{as } |x| \to \infty. \tag{4}$$

The appropriate fundamental solution is

$$G(P, Q) = K_0(lR) + \int_0^\infty \cos[l(x-\xi)\sinh\mu]\,\frac{\cosh\mu + \cos\beta}{\cosh\mu - \cos\beta}\,e^{-l(y+\eta)\cosh\mu}\,d\mu, \tag{5}$$

where

$$P = (x, y),\quad Q = (\xi, \eta),$$
$$R = ((x-\xi)^2 + (y-\eta)^2)^{\frac{1}{2}}$$

and K_0 is the modified Bessel function of the second kind, of order zero. The fundamental solution satisfies (1), (2) and (4) and has a logarithmic singularity at $P = Q$. If we now apply Green's theorem to ϕ and G, we find

$$\phi(P) = \int_{\Gamma} [\phi(q)] \frac{\partial G(P,q)}{\partial n_q} ds_q, \tag{6}$$

where P is any point in the fluid, and $[\phi(q)]$ represents the discontinuity in potential across the plate. Imposing (3) to (6), and interchanging integration and normal differentiation, gives

$$⨎_{\Gamma} [\phi(q)] \frac{\partial^2 G(p,q)}{\partial n_p \partial n_q} ds_q = 0 \qquad p \in \Gamma, \tag{7}$$

where the integral must now be interpreted as a Hadamard finite–part integral; see [3] and [4] for further information on hypersingular integral equations. Our aim is to find pairs of parameters l and β for which non-trivial solutions to (7) exist.

3 Method of Solution

For numerical computations, we use the expansion, see Ursell [5],

$$G(P,Q) = K_0(lR) + K_0(lR_1)$$
$$+2\cot\beta e^{-l(y+\eta)\cos\beta} \{(\pi-\beta)\cosh[l(x-\xi)\sin\beta] - \Theta_1 \sinh[l(x-\xi)\sin\beta]\}$$
$$-4\cot\beta \sum_{m=1}^{\infty} (-1)^m L_m(lR_1)\cos m\Theta_1 \sin m\beta,$$

where $x - \xi = R_1 \sin\Theta_1$, $y + \eta = R_1 \cos\Theta_1$ and

$$L_m(lR_1) = \left(\frac{\partial}{\partial\nu} I_\nu(lR_1)\right)_{\nu=m}.$$

This expansion for G may be easily differentiated to find the kernel of (7). To seek a numerical solution to (7), we first parametrise Γ; we may take the length of the plate as 2, without losing any generality. This gives us

$$⨎_{-1}^{1} \frac{f(t)}{(s-t)^2} dt + \int_{-1}^{1} f(t) L'(s,t)\, dt = 0,$$

where $L'(s,t)$ also depends on the angle that the plate makes with the vertical, and on d, the submergence of the mid-point of the plate. We have also replaced $[\phi(q)]$ by $f(t)$. We find that $L'(s,t)$ has a logarithmic singularity at $s = t$. More precisely, we have

$$L'(s,t) \sim \frac{l^2}{2} \ln|s-t| \qquad \text{as } |s-t| \to 0.$$

Thus, we write

$$⨎_{-1}^{1} \frac{f(t)}{(s-t)^2} dt + \frac{l^2}{2} \int_{-1}^{1} f(t) \ln|s-t|\, dt + \int_{-1}^{1} f(t) L(s,t)\, dt = 0,$$

where $L(s,t)$ is now well behaved as $|s-t| \to 0$. As with scattering problems, we now approximate $f(t)$ with the series

$$f(t) \cong \sqrt{1-t^2} \sum_{n=0}^{N} a_n U_n(t),$$

where $U_n(t)$ is the nth Chebyshev polynomial of the second kind. The unknown coefficients, a_n, are then found by collocation. This choice of approximation has two attractive properties. First, the square–root zeros in $[\phi]$ at the plate edges have been built into the solution. Also, the integrals containing the hypersingular and logarithmic kernels can be evaluated exactly. We have,

$$f_n(s) = ⨎_{-1}^{1} \frac{\sqrt{1-t^2}U_n(t)}{(s-t)^2} dt = -\pi(n+1)U_n(s)$$

and

$$g_n(s) = \int_{-1}^{1} \sqrt{1-t^2}\,U_n(t)\ln|s-t|\,dt = \begin{cases} \frac{\pi}{2}\left(-\ln 2 + \frac{\cos 2\psi}{2}\right) & \text{when} \quad n=0 \\ \frac{\pi}{2}\left(\frac{\cos(n+2)\psi}{n+2} - \frac{\cos n\psi}{n}\right) & \text{when} \quad n \geq 1, \end{cases}$$

where $s = \cos\psi$. In the evaluation of this last integral, we have used the expansion

$$\ln|s-t| = -\ln 2 - 2\sum_{n=1}^{\infty} \frac{1}{n} T_n(s)T_n(t),$$

where T_n is the nth Chebyshev polynomial of the first kind. The integrals involving $L(s,t)$ may now all be evaluated numerically. The only slight difficulty encountered in this method is the evaluation of $L_m(x)$. We adopt the following scheme. The modified Bessel function, $I_\nu(x)$, satisfies the homogeneous recurrence relation

$$I_{\nu-1} - I_{\nu+1} - \frac{2\nu}{x} I_\nu = 0,$$

whence differentiation with respect to ν, and letting $\nu = m$, shows that $L_m(x)$ satisfies the second order, nonhomogeneous recurrence relation

$$L(m-1) - L(m+1) - \frac{2m}{x} L(m) = \frac{2}{x} I(m). \tag{8}$$

Here, for ease of notation, we write the order as the argument. To solve (8), we use the Wimp–Luke method, see Wimp [6], as follows. We generate a solution $Z_M(m)$ of the nonhomogeneous equation, in the *backwards* direction, with starting values

$$Z_M(M+1) = Z_M(M) = 0.$$

We then generate a solution $Y_M(m)$ of the homogeneous equation; that is, we take $I(m) \equiv 0$. This solution is also found in the backwards direction, with the starting values

$$Y_M(M+1) = 0 \text{ and } Y_M(M) = 1.$$

The solution to (8) is then formed by letting

$$L_M(m) = \left(\frac{L(0) - Z_M(0)}{Y_M(0)}\right) Y_M(m) + Z_M(m), \quad 0 \leq m \leq M+1,$$

where $L(0) = -K_0$. In order to achieve greater accuracy, we can iterate; that is, we take $L_M(M)$ and $L_M(M-1)$ as new starting values $Z_M(M)$ and $Z_M(M-1)$, and then repeat the above procedure. This process may be repeated again, until no further improvement is achieved. In practice, provided M is chosen slightly greater than the maximum value of m required in the summation containing L_m, and only one iteration is performed, the results were found to be excellent.

Returning to the problem of finding the trapped-mode frequencies, we have the system of equations, $\mathbf{Ka} = 0$, to solve, where

$$K_{jn} = f_n(s_j) + \frac{l^2}{2} g_n(s_j) + \int_{-1}^{1} \sqrt{1-t^2}\,U_n(t) L(s_j,t)\,dt, \qquad j,n = 0,1,\ldots,N,$$

$\mathbf{a} = (a_0, a_1, \ldots, a_N)^T$ and s_j are the collocation points. The trapped-mode frequencies are now given by $\det(\mathbf{A}) = 0$, provided that N is chosen sufficiently large to ensure convergence. To look for these frequencies, we first seek an approximation to them, so that the numerical search may be speeded up.

4 Approximate Solutions

For a *horizontal* plate, of length $2a$, we may use a similar method to that used by Linton & Evans [2] in approximating the desired frequencies, with some modifications to account for deep water. The method uses a wide–spacing approximation. As in the paper mentioned above, we assume that a wave, with $\exp(ilz)$ dependence, is incident from $x = -\infty$ along a semi–infinite plate, lying on $y = d, x < 0$. For values of l in a certain range, total reflection occurs at the edge of the plate. The reflection coefficient is related to l, through the equation

$$R = \exp(-2i\alpha a), \tag{9}$$

where $\alpha = (k^2 - l^2)^{\frac{1}{2}}$ and k solves $K = k \tanh kd$. An explicit expression for R can be found by use of the Wiener–Hopf technique; a similar problem with the same geometry was solved by Greene & Heins [1]. Comparison of this result with (9), then gives approximations to the trapped–mode frequencies.

We intend to compare the wide–spacing approximation with the 'exact' numerical results, for a horizontal plate. Having done this, the trapped–mode frequencies can be followed as the inclination of the plate is varied.

References

[1] Greene, T.R. & Heins, A.E. Water waves over a channel of infinite depth. *Quart. of Appl. Maths.* **2** (1953) 201–214.

[2] Linton, C.M. & Evans, D.V. Trapped modes above a submerged horizontal plate. *Quart. J. Mech. Appl. Math.* 44 (1991) 487–506.

[3] Parsons, N.F. & Martin, P.A. Interaction of water waves with thin plates. *Proc. 7th International Workshop on Water Waves & Floating Bodies.* Val de Reuil. France (1992) 227–231.

[4] Parsons, N.F & Martin, P.A. Scattering of water waves by submerged plates using hypersingular integral equations. *Appl. Ocean Research.* 14 (1992) 313–321.

[5] Ursell, F. Slender oscillating ships at zero forward speed. *J. Fluid Mech.* 14 (1962) 496–516.

[6] Wimp, J. *Computation with recurrence relations.* Pitman, London (1984) 137–139.