An existence theorem for trapped modes in channels

by

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Introduction

The existence of trapped modes above a long submerged horizontal cylinder of sufficiently small radius, in deep water, was proved by Ursell (1951) over forty years ago. However it was not until recently that evidence emerged for the existence of trapped modes in the vicinity of a *vertical* cylinder extending throughout the water depth and mid-way between the walls of a channel of infinite extent. Thus Evans and Linton (1991) provided numerical evidence for the existence of such modes, antisymmetric about the mid-plane of the channel, for the case of a rectangular cylinder having two opposite faces parallel to the channel walls. Again Callan, Linton and Evans (1991), using methods similar to Ursell (1951) proved that an antisymmetric trapped mode existed for a sufficiently small cylinder of circular cross-section whilst Evans (1992) proved their existence for sufficiently long thin vertical plates positioned parallel to and mid-way between the walls of the channel. Finally Linton and Evans (1992) used an appropriate Green's function to construct a homogeneous integral equation for the trapped modes in the case of a cylinder of fairly general cross-section and showed that the trapped mode frequencies agreed numerically with the previous results for the cylinder and rectangular cross-sections.

In this paper we consider the general question of the existence of antisymmetric trapped modes in the vicinity of a vertical cylinder of fairly arbitrary cross-section extending throughout the water depth and placed mid-way between channel walls of infinite extent.

Formulation

We choose Cartesian coordinates and eliminate the depth variation by the factor $\cosh k(z+h)$ where h is the channel depth and $\omega^2 = gk \tanh kh$. Then the channel is described by $G = \{(x,y) : |y| < d\}$ with the walls represented by the parallel lines $\Gamma_\pm = \{y = \pm d\}$. The cylinder cross-section F has boundary $\Phi = \Phi_+ \cup \Phi_-$ assumed to be piecewise smooth and parametrized by $\Phi_\pm = \{(x,y) : x = X(s), y = \pm Y(s), 0 \le s \le L, X_s'^2 + Y_s'^2 = 1, 0 \le Y(s) < d, Y(0) = Y(L) = 0, X(0) = -a, X(L) = a\}$. The only restriction on the shape of the cylinder is $X_s' \ge 0$. We denote by $2a = X(L) - X(0)$ the diameter of the cylinder, and by $S(X)$ the inverse function to $X(s)$ so that $x = X(S(x))$ where $S(x)$ is unique; where $X'(s) = 0$ we define $S(x) = \min\{s : x = X(s)\}$. Finally we assume for simplicity that $Y(s) \not\equiv 0$. Thus the case of the thin plate requires a modification to the proof.

In considering trapped modes we shall seek solutions $\phi(x,y)$ which are antisymmetric about the axis $y=0$ and satisfy $\phi(x,y) = -\phi(x,-y)$. Then we need only consider $0 \le y \le d$ and $\phi(x,y)$ satisfies

$$\begin{aligned}
(\nabla^2 + k^2)\phi &= 0 \quad \text{for } (x,y) \in V_+ \equiv G_+ \backslash F_+ && (1)\\
\frac{\partial \phi}{\partial y} &= 0 \quad \text{for } (x,y) \in \Gamma_+ && (2)\\
\frac{\partial \phi}{\partial n} &= 0 \quad \text{for } (x,y) \in \Phi_+ && (3)\\
\phi &= 0 \quad \text{for } (x,y) \in \Phi_0 && (4)\\
\phi &\to 0 \quad \text{for } |x| \to \infty, 0 \le y \le d. && (5)
\end{aligned}$$

Here $G_+ = \{(x,y) : 0 < y < d\}$, F_+ is one half of the cylinder F, V_+ is the region occupied by the fluid and $\Phi_0 = \{(x,y) : y = 0, x \in (-\infty, a] \cup [a, \infty)\}$.

We seek solutions of (1)–(5) which are infinitely differentiable at all points of V_+ and up to the smooth parts of the boundary, and continuous at all the 'corners' of Φ_+. Such a solution is a trapped mode and k^2 is then an eigenvalue which belongs to the discrete spectrum of the problem defined above, which we denote by (P). If there exist non-trivial solutions to (1)–(4) which do *not* vanish as $|x| \to \infty$ we say the corresponding value of k^2 belongs to the continuous spectrum. The following result is well-known:

Lemma 1 *The continuous spectrum of (P) is the semi-interval* $[\pi^2/4d^2, \infty)$.

Physically Lemma 1 means that for any positive $k > \pi/2d$ there exists a mode of vibration radiating to $x = \pm\infty$, but no such modes exist for $k < \pi/2d$. We shall prove that there is at least one value of $k < \pi/2d$ satisfying (P).

We first introduce the space C^∞ of functions $\psi(x,y)$ which are infinitely differentiable at all points of V_+ and on to the boundary. By $\tilde{C}_0^\infty$ we denote the subspace of C^∞ consisting of functions $\psi(x,y)$ satisfying the following two properties:

$$\begin{aligned}
\psi(x,0) &= 0 \quad \text{for } (x,y) \in \Phi_0 && (6)\\
\psi(x,y) &= 0 \quad \text{for sufficiently large } |x|. && (7)
\end{aligned}$$

Then we have the following fundamental variational principle for (P). (For a general formulation, see, for instance, Birman and Solomjak (1986)).

Lemma 2 *Let*

$$k_0^2 = \inf \iint_{V_+} |\text{grad}\psi|^2 \, dx dy \Big/ \iint_{V_+} |\psi|^2 \, dx dy$$

where the infimum is sought amongst functions in $\tilde{C}_0^\infty$. *If* $k_0^2 < \pi^2/4d^2$ *then* k_0^2 *is the smallest eigenvalue of (P).*

Let us choose a smooth cut-off function $\chi(x)$ with the properties

$$\begin{aligned}
\chi(x) &= 1, & |x| &\le 1\\
0 < \chi(x) &< 1, & 1 < |x| &< 2\\
\chi(x) &= 0, & |x| &\ge 2
\end{aligned}$$

and define the function $\Psi(x,y) = \sin(\pi y/2d)\chi(x/A)$ where $A > 0$. Clearly $\Psi \in \tilde{C}_0^\infty$. We can now prove

Lemma 3 *For sufficiently large A,*

$$\iint_{V_+} |\mathrm{grad}\psi|^2 \,\mathrm{d}x\mathrm{d}y < \frac{\pi^2}{4d^2} \iint_{V_+} |\psi|^2 \,\mathrm{d}x\mathrm{d}y$$

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The proof is elementary but laborious, so is omitted here. Lemmas 2 and 3 immediately imply $k_0^2 < \pi^2/4d^2$, and it is easily shown that $k_0^2 \neq 0$ from which we conclude:

Theorem. *There exists a trapped mode $k_0 \in (0, \pi/2d)$ of (P).*

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