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Numerical Study of Three-Dimensional Overturning Water Waves

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The kinematics of three-dimensional breaking waves is of fundamental importance to theoretical understanding and engineering applications. Because of mathematical difficulties in solving the nonlinear wave equations in three dimensions, three-dimensional overturning waves are among the least well studied wave phenomena. In particular, weakly nonlinear theories based on perturbations are of limited use because of the non-single-valued nature of the overturning wave profile.

In this paper, we present a three-dimensional extension of the numerical study of periodic two-dimensional overturning waves by Longuet-Higgins & Cokelet (1976) using a mixed-Eulerian-Lagrangian (MEL) approach. We consider an inviscid, incompressible and irrotational free surface flow on deep water with spatial periodicity in both horizontal x and y directions. We choose a computation domain of length L and width W. The Cartesian coordinate system is defined with its origin in the undisturbed water level and the z axis positive upward. For simplicity, all variables are nondimensionalized by setting the density of water ρ and acceleration due to gravity g to unity.

The potential ϕ satisfies the Laplace equation $\nabla^2\phi = 0$ with appropriate boundary conditions. On the free surface S_f, Bernoulli's equation gives the dynamic boundary condition:

$$\frac{D\phi}{Dt} = \frac{1}{2}|\nabla\phi|^2 - z - p_f, \qquad \vec{x} \in S_f, \tag{1}$$

where $D/Dt = \partial/\partial t + \nabla\phi\cdot\nabla$ is the material derivative following a Lagrangian particle, and $p_f(\vec{x},t)$ the (applied) surface pressure. The kinematic boundary condition is

$$\frac{D\vec{x}}{Dt} = \nabla\phi, \qquad \vec{x} \in S_f. \tag{2}$$

This vector equation gives the evolution equation for free-surface Lagrangian particles. For deep water, the appropriate far-field condition is:

$$\nabla\phi(\vec{x}) \to 0 \qquad \text{as } z \to -\infty, \tag{3}$$

for all times.

Assuming, without loss of generality, that ϕ is a constant at infinity, we obtain from Green's second identity

$$\iint_{S_f} \phi_n G_p dS + \iint_{S_\infty} \phi G_{p,n} dS = -\alpha\phi - \iint_{S_f} \phi G_{p,n} dS \tag{4}$$

ϵ	N_w	u		w		ϕ_n	
		$\bar{e}$	e_{max}	$\bar{e}$	e_{max}	$\bar{e}$	e_{max}
0.4	8	0.02463	0.05460	0.01165	0.02547	0.01294	0.04134
	16	0.00271	0.00914	0.00581	0.02914	0.00560	0.02453
	32	0.00027	0.00094	0.00126	0.00640	0.00090	0.00503
	64	0.00010	0.00045	0.00055	0.00238	0.00046	0.00197

Table 1: Error convergence of QBEM for an exact Stokes wave profile with $L = 2\pi$ ($W = L/4$). Curvature based grids are used. H_w is the number of unknowns per wavelength. u and w designate the velocity components in x and z. $\bar{e}$ and e_{max} denote the arithmetically averaged and maximum errors in the computed quantities, respectively.

where $\vec{x} \in S_f$ and α is the solid angle at $\vec{x}$, and the finite part of an integral is assumed if the kernel is singular. In the above, G_p is the doubly-periodic (in x and y) harmonic Green function with the following far field behaviour

$$G_p = C|z| + o(1) \tag{5}$$

where C is a finite constant. Clearly, G_p does not satisfy the far-field condition (3). Since $G_{n,p}$ is a constant at S_∞, the far-field integral in (4) evaluates to a constant, τ, say. A compatibility (Gauss) condition

$$\iint_S \phi_n(\vec{x}, t) dS = 0 \tag{6}$$

gives the additional equation for τ.

After extensive testing and comparisons with other boundary-integral approaches, we select and extend a high-order boundary-integral equation method based on bi-quadratic isoparametric curvilinear elements (QBEM). The regular part of G_p is evaluated effectively using expansions given by Newman (1991). The singular integrals over a curvilinear domain are regularized by triangular polar-coordinate transformations in the parametric space. An efficient adaptive quadrature scheme is developed for the elemental integrals. For far-field collocation points, efficient series expansions are adopted in favor of numerical quadrature. The resulting linear system of equations is solved using the GMRES iterative scheme (Saad & Schultz 1986) with a SSOR preconditioner. Our numerical experiments confirm that for Kellogg-regular boundaries, both the maximum and average error of QBEM exhibit quadratic convergence with the number of unknowns. Table 1 shows the typical convergence of these errors for a test case using exact Stokes waves.

To advance Lagrangian points in the present MEL context, the accuracy of the solved velocity field is crucial. For the computation of the (tangential) velocity components on the nonlinear free surface in three dimensions, a parametric finite-difference scheme (Xü 1992) is devised, which is superior in efficiency and accuracy to the use of bi-cubic spline fitting. For the time integration of the nonlinear free-surface conditions, a fourth-order Adams-Bashforth-Moulton difference formula is used with a fourth-order Runge-Kutta scheme as a starting procedure. The time step size is dynamically controlled by stability criteria based on particle velocity and panel dimensions (Dommermuth, Yue *et al* 1988). To suppress the growth of saw-tooth instabilities, a five-point Chebyshev smoothing formula is applied in alternating directions in the parametric space after a fixed integral number of time steps.

Systematic accuracy and convergence tests are performed first using exact finite-amplitude Stokes waves, and in repeating the two-dimensional overturning wave simulations using an applied pressure of Longuet-Higgins & Cokelet (1976) and others. The results are in generally excellent agreement.

To generate a three-dimensional overturning wave, we start with a progressive two-dimensional Stokes wave but now apply a *three-dimensional* surface pressure distribution to raise the energy density beyond the maximum for a steady Stokes wave. Specifically, we choose the following form for the surface forcing:

$$p_f = \begin{cases} p_0[1+\cos(2\pi y/W)]\sin t \sin(x-ct) & \text{for } 0 \le t \le \pi, \\ 0 & \text{for } t > \pi, \end{cases} \tag{7}$$

with $p_0 = 0.073$, and the wavelength of the Stokes wave is $L{=}2\pi$. To quantify the degree of three-dimensionality, three different values of the (periodic) transverse width are chosen corresponding to $W/\pi{=}1$, 2 and 3. For computational efficiency, only half of the symmetric domain in y is discretized in the simulations. Figure 1.1 shows the perspective view of the near-final phases of the overturning wave profiles corresponding to the three different W values. (Only the symmetric halves corresponding to $y \in [-W/2, 0]$ are shown. The profile nearest the observation point is along $y = -W/2$ and the furtherest profile is the symmetry line $y = 0$.)

Although the maximum of the forcing pressure for all three cases are along the centerline $y = 0$, the resulting three-dimensional plunging breakers surprisingly develop either at the center ($y \sim 0$) or at the edges ($y \sim \pm W/2$) depending on the value of W/L. Equally interesting are the time histories of the kinetic and potential energies, which after normalization by the width W, differ only by a few percent so that the total energies are approximately linear functions of W. It is noteworthy, however, that the velocity/acceleration fields and profiles of these three-dimensional overturning waves are otherwise quite different and qualitatively so for $W \ge L$ versus $W < L$.

More detailed numerical results and discussions of the kinematics of three-dimensional overturning waves will be presented at the Workshop.

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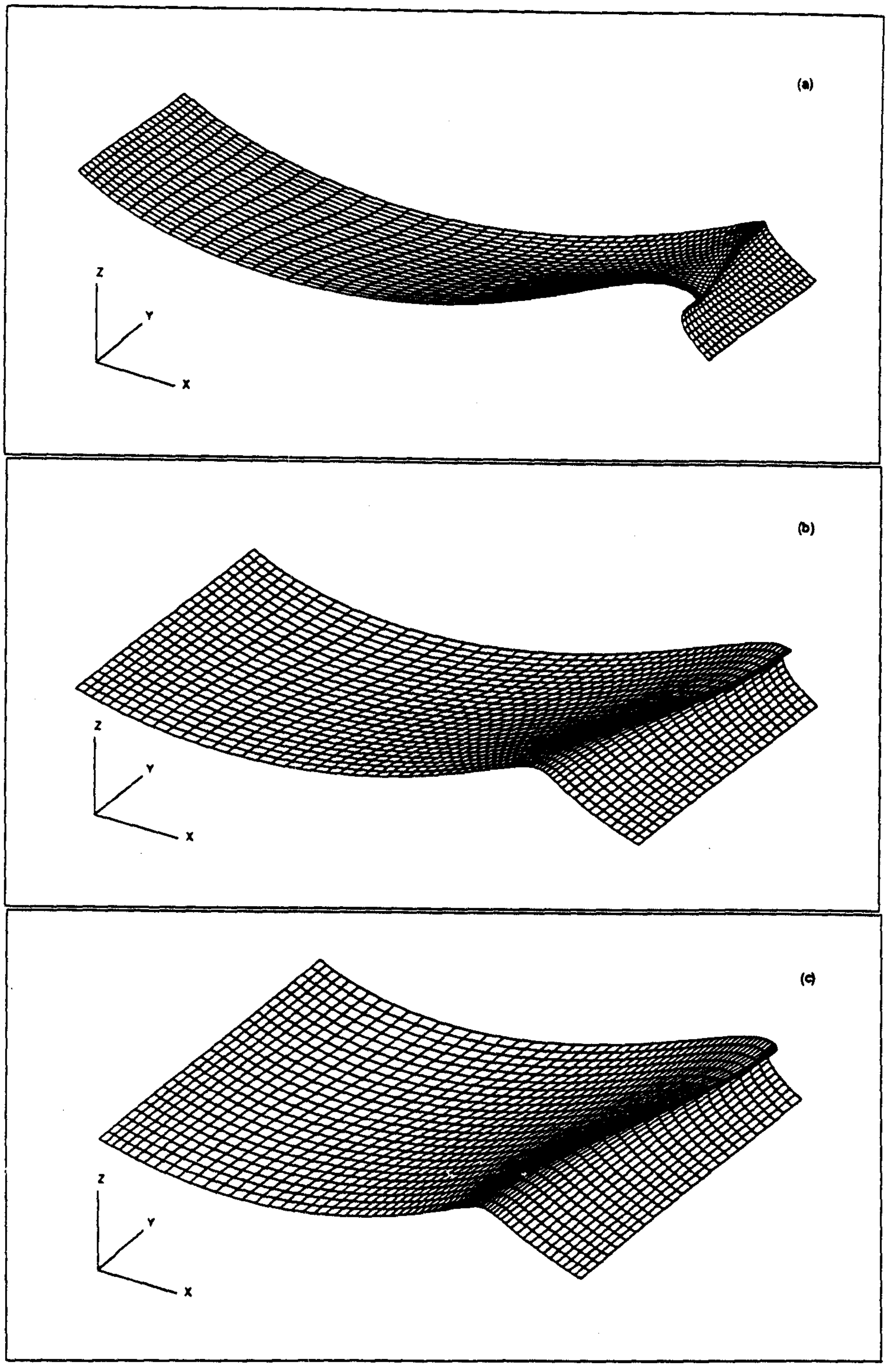

Figure 1.1: A perspective view of the overturning waves for (a) $W = \pi$, at $t = 5.25$ (b) $W = 2\pi$, at $t = 4.87$ (c) $W = 3\pi$ at $t = 4.82$.

DISCUSSION

GREENHOW: What type of ocean engineering application is now open to you as a result of this 3D approach? By the way, this is really impressive work!

XÜ & YUE: Our long term goal is the simulation of general non-linear wave-body problems such as the large-amplitude motions and loads on a ship in steep waves. This capability is still some time away, and requires, among other developments, a robust geometry capability involving three-dimensional surfaces and interactions. More immediate applications include interaction of waves with submerged obstacles, 3D nonlinear developments of wavefields, sloshing in tanks and diffraction by vertical cylinders/struts.

TUCK: Why is the Lagrangian method still being used? It has always seemed to me quite unnatural to formulate the problem in an Eulerian manner but to solve it in a Lagrangian manner. There were some advantages of convenience when Languet-Higgins and Cokelet began this type of numerical study, but with the advance of computational skills and hardware power, I should have thought that a fully Eulerian method would have replaced the Lagrangian method by now. In detail, I have always been uneasy about the fact that at each step one solves a ***Dirichlet*** problem on the free surface, which bears no relationship to the true physical situation.

XÜ & YUE: It seems to us that your question touches on 2 aspects:

(1) why update following ***Lagrangian*** points rather than in an Eulerian manner? The use of Lagrange points is neither essential nor critical in integrating the free surface in time — (a) for non single value free surface F, there is no 'obvious' projection with which to specify Eulerian updating and Lagrangian point update is both natural and simple; (b) it is known, that Lagrangian points have a tendency to concentrate in regions of rapid variation, so following Lagrangian points leads to enhanced numerical resolution.

(2) why specify the 'unnatural' Dirichlet free-surface boundary condition? This we believe may be the main point of your question. For free-surface problems, the physically ***natural*** boundary condition to specify on F is in fact the Dirichlet one, i.e., at (initial) time t, we specify the position F(t) and the pressure (impulse) $\phi(t)$ on F(t) and seek the subsequent evolution (solve for $\phi_n(t)$ and thus $\nabla\phi(t)$ on F(t), then use the kinematic and dynamic conditions to obtain respectively $F(t+\Delta t)$ and $\phi(t+\Delta t)$ on $F(t+\Delta t)$, and the process is repeated). This 'natural' specification unfortunately leads to mathematically undesirable ('unnatural') first kind boundary-integral equations (at least for the direct formulation in terms of ϕ and ϕ_n using Green's theorem).The alternative is to specify $\phi_n(t)$ on F(t), which, after solving (2nd king equations) for $\phi(t)$ and hence $\nabla\phi(t)$ on F(t), does not allow us a natural way to determine $\phi_n(t+\Delta t)$!