THE SECOND ORDER WAVE LOADS ON BODIES IN STRATIFIED OCEAN

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In stratified ocean, the internal waves may be important in prediction of the wave loads and responses of floating bodies. Some efforts had taken towards investigating the stratification effects upon the wave loads and responses of constrained floating bodies. Wu *et al* had developed a linear theory and found that, the first and second order equations of motion for a constrained floating body are similar to those in uniform ocean[1], the added mass and damping matric both also are symmetric, there exist two radiation wave systems with wave numblers $\{k_{1n}\}$ $(n = 1, 2, \cdots, M)$ at far field, where k_{11} is the free surface wave numbler and others the internal wave numblers[2], and the effects of stratification should be considered when estimating the low frequency hydrodynamic characters of a floating body[3]. For a floating vertical circular cylinder in two layer fluids of different densities, their numerical results[4] indicated that, if the body is in the upper layer fluid, the effects of stratification upon the hydrodynamic coefficients become more significant when the body more near the interface of the two fluids.

Here, We consider the second order wave diffraction problem of a fixed body in stratified ocean. It is assumed that the seawaters consist of two stationary superposed fluids with different density ρ_1 and ρ_2 , where ρ_1 is the density of the upper layer water and ρ_2 the density of the lower layer water, the fluids are imcompressible, and the motion is irrotational. Approximate a random seaway by the linear surperposition of a sufficiently large number of regular plane progressive wave components of different frequencies $\{\omega_n\}$ and headings $\{\alpha_n\}$, and denote the linear order incident and scattering potential by

$$Re\,(\varphi^I_m(x)exp(-i\omega_m t)) \quad \text{and} \quad Re(\varphi^S_n(x)exp(-i\omega_m t)),$$

respectively, one can find the second order scattering potential Φ^S_2 has following expression:

$$\Phi^S_2(x,t) = \sum_{m,n=1} Re(\Phi^{\pm}_{nm}(x)exp(-i\Omega^{\pm}_{nm}t) + \Phi^{-}_{nm}(x)exp(-i\Omega^{-}_{nm}t)) \tag{1}$$

where, $\Omega^{\pm}_{nm} = \omega_n \pm \omega_m$. The sum- and difference- frequency potentials $\Phi^{\pm}_{nm}$ and Φ^{-}_{nm} are governed by

$$\nabla^2\Phi^{\pm}_{nm} = 0 \qquad x_3 \in (-H,-h) \cup (-h,0) \tag{2}$$

$$L^{\pm}_{nm}\Phi^{\pm}_{nm} = p^{\pm}_{nm}(x) \qquad \text{on } x_3 = 0 \tag{3}$$

$$\rho_1 L^{\pm}_{nm}\Phi^{\pm}_{nm}(x_1,x_2,-h+o) - \rho_2 L^{\pm}_{nm}\Phi^{\pm}_{nm}(x_1,x_2,-h-o) = U^{\pm}_{nm}(x_1,x_2) \tag{4}$$

$$\partial_3\Phi^{\pm}_{nm}(x_1,x_2,-h+o) - \partial_3\Phi^{\pm}_{nm}(x_1,x_2,-h-o) = V^{\pm}_{nm}(x_1,x_2) \tag{5}$$

$$\partial_3\Phi^{\pm}_{nm} = 0 \qquad \text{on } x_3 = -H; \tag{6}$$

$$n\nabla(\Phi^{\pm}_{nm} + \varphi^{\pm}_{nm}) = 0 \qquad \text{on } x \in S_B \tag{7}$$

and a proper radiation condition at far field. Where, H, the sea bottom depth, is constant, n the unit normal vector pointing into the body, S_B the wetted surface of the body, x_3 axis the vertical axis, positive upward, $x_3 = 0$ corresponds to the mean free surface, and $x_3 = -h$ expresses the mean interface between the two superposed fluids. φ^{+}_{nm} and φ^{-}_{nm} are the second order sum- and difference-incident potential, respectively, $L^{\pm}_{nm} = g\partial_3 - (\omega_n \pm \omega_m)^2$, $x = (x_1,x_2,x_3)$, $\partial_j = \partial/\partial x_j (j = 1,2,3)$. The forcing term $P^{\pm}_{nm}$, $\Omega^{\pm}_{nm}$ and $V^{\pm}_{nm}$ can be expressed as

$$2P^{\pm}_{nm}(x) = i\Big(\Omega^{\pm}_{nm}\nabla\varphi^S_n\nabla\varphi^{S,\pm}_m - \omega_n\varphi^S_n L_m\varphi^{S,\pm}_m$$

$$+ 2\Omega^{\pm}_{nm}\nabla\varphi^S_n\nabla\varphi^{I,\pm}_m - \omega_n\varphi^S_n L_m\varphi^{I,\pm}_m \mp \omega_m\varphi^{I,\pm}_m L_n\varphi^S_n\Big) \tag{8a}$$

$$U^{\pm}_{nm}(x_1,x_2) = \rho_1 P^{\pm}_{nm}(x_1,x_2,-h+o) - \rho_2 P^{\pm}_{nm}(x_1,x_2,-h-o) \tag{8b}$$

$$2V^{\pm}_{nm}(x_1,x_2) = \nabla_1(\eta^S_n\nabla_1\varphi^{S,\pm}_m) + \nabla_1(\eta^S_n\nabla_1\varphi^{I,\pm}_m) + \nabla_1(\eta^{I,\pm}_m\nabla_1\varphi^S_n) \tag{8c}$$

where

$$\eta^S_n = i\omega_n^{-1}\partial_3\varphi^S_n(x_1,x_2,-h), \quad \eta^I_n = i\omega_n^{-1}\partial_3\varphi^I_n(x_1,x_2,-h)$$

$$\nabla_1 = (\partial_1,\partial_2), \qquad L_m = \partial_3^2 - v_m\partial_3, \qquad v_m = \omega_m^2/g$$

$$\varphi^{I,+}_m = \varphi^I_m = (\varphi^{I,-}_m)^*, \qquad \varphi^{S,+}_m = \varphi^S_m = (\varphi^{S,-}_m)^*.$$

The Linear incident potential φ^I_n is given as[1]

$$\varphi^I_n = i\omega_n^{-1}a_n f(x_3)\cosh k_n(x_3+H)\cosh k_n H exp[i(k_{n1}x_1 + k_{n2}x_2 - \omega_n t + \delta_n)]$$

$$f(x_3) = \begin{cases} 1 - v_n T + (v_n - T)\tanh k_n(x_3+H) & x_3 \in (O,-h) \\ 1 - v_n T + (v_n - T)T_2 & x_3 \in (-h,-H) \end{cases} \tag{9}$$

where $T = \tanh k_n H$, $T_1 = \tanh k_n h$, $T_2 = \tanh k_n(H-h)$, and the wave number k_n, which equals to $(k_{n1}^2 + k_{n2}^2)^{1/2}$, is one of the positive real roots of the dispersion equation as follows

$$\frac{\omega_n^2}{k_n} = \frac{rT(1+T_1T_2) \pm \sqrt{r^2T^2(1+T_1T_2)^2 - 4(r-1)T_1T_2(r+T_1T_2)}}{2(r+T_1T_2)} \tag{10}$$

with $r = \rho_1/\rho_2$, and, in connection with the source distribution method, the solution of the linear scattering potential φ_n^S had been obtained and can be expressed as[3]

$$\varphi_n^S(x) = \int_{S_B} \sigma_n(q) G(x,q;v_n) dS, \quad q = (q_1,q_2,q_3) \in S_B \tag{11}$$

where, σ_B is the source density, and the Green function G is the solution of below boundary value problem:

$$\nabla^2 G(x,q;v_n) = \delta(x-q) \qquad q_3,x_3 \in (-H,-h) \cup (-h,O) \tag{12}$$

$$L_n G = 0, \quad \text{on } x_3 = 0; \qquad \partial_3 G = 0, \quad \text{on } x_3 = -H \tag{13}$$

$$\rho_1 L_n G(x_1,x_2,-h+o) = \rho_2 L_n G(x_1,x_2,-h-o) \tag{14}$$

$$\partial_3 G(x_1,x_2,-h+o) = \partial_3 G(x_1,x_2,-h-o) \tag{15}$$

Sommerfeld condition at far field (16)

it's integral expression had derived from [3].

The far field condition of the second order problem also can be derived along the same way described by Wu et. al[5].

We consider the general case of a surface-piercing body, and, for simplicity, assume that the body surface penetrates the free surface vertically. Thus, the second order force $F_{nm\alpha}$ on the body has following expression:

$$F^{\pm}_{nm\alpha} = F^{\pm}_{1nm\alpha} + F^{\pm}_{2nm\alpha} \tag{17}$$

$$F^{\pm}_{1nm\alpha} = \frac{1}{8}\Big(\int_{S_B} \rho\nabla(\varphi_m^I + \varphi_m^S) \cdot \nabla(\varphi_n^{I,\pm} + \varphi_n^{S,\pm}) n_\alpha dS$$

$$\pm \rho_1 \omega_n \omega_m \int_{L_{1W}} (\varphi_m^I + \varphi_m^S)(\varphi_n^{I,\pm} + \varphi_n^{S,\pm}) n_\alpha dL + c.c\Big) \tag{18}$$

$$F^{\pm}_{2nm\alpha} = -i\frac{1}{2}(\omega_n \pm \omega_m)\int_{S_B} \rho(\varphi^{\pm}_{nm} + \Phi^{\pm}_{nm}) n_\alpha dS + C.C \tag{20}$$

where S_B is the wetted body surface below the still water level $x_3 = 0$, and L1w is the waterline curve at $x_3 = 0$.

Introducing an adjacent function $\psi^{\pm}_{nm\alpha}(x)$ governed by

$$\nabla^2 \psi^{\pm}_{nm\alpha} = 0 \qquad x_3 \in (-H,-h) \cup (-h,0) \tag{21}$$

$$L^{\pm}_{nm} \psi^{\pm}_{nm\alpha} = 0 \qquad \text{on } x_3 = 0 \tag{22}$$

$$\rho_1 L^{\pm}_{nm} \psi^{\pm}_{nm\alpha}(x_1,x_2,-h+o) - \rho_2 L^{\pm}_{nm} \psi^{\pm}_{nm\alpha}(x_1,x_2,-h+o) = 0 \tag{23}$$

$$\partial_3 \psi^{\pm}_{nm\alpha}(x_1,x_2,-h+o) - \partial_3 \psi^{\pm}_{nm\alpha}(x_1,x_2,-h-o) = 0 \tag{24}$$

$$\partial_3 \psi^{\pm}_{nm\alpha} = 0 \qquad \text{on } \chi_3 = -H; \tag{25}$$

$$n \nabla \psi^{\pm}_{nm\alpha} = n_\alpha \qquad \text{on } \chi \in S_B \tag{26}$$

and Sommerfeld condition at far field, and, using Green's theorem, we find that, $F^{\pm}_{2nm\alpha}$, the part of the second order wave forces which is only in connection with the forcing terms on the boundary condition of the mean free surface and the interface of the two fluids, can be rewritten as

$$
\begin{aligned}
&\int_{S_B} \rho \Phi^{\pm}_{nm} n_\alpha dS \\
&= \rho_1 \int_{S_F} \psi^{\pm}_{nm\alpha}(x) P^{\pm}_{nm}(x) dS \\
&+ \int_{S_I} \Big((\omega_n \pm \omega_m)^{-2} \partial_3 \psi^{\pm}_{nm\alpha}(x) U^{\pm}_{nm}(x) - \frac{\rho_1 \rho_2}{\rho_1 - \rho_2} V_{nm}(x_1, x_2) \\
&\times [\psi^{\pm}_{nm\alpha}(x_1, x_2, -h+o) + \psi^{\pm}_{nm\alpha}(x_1, x_2, -h-o)] \Big) dS
\end{aligned}
\tag{27}
$$

where S_I is the mean interface of the two fluid layer.

The practical calcutation of the second order wave forces is our next task. This work is supported by the Chinese National Naturel Science Foundation (Grant 18902011)

Reference

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DISCUSSION

GREENHOW: This problem may be important for a floating bridge structure proposed for crossing Norwegian fjords, where strong (fresh/salt) density stratification can occur. I suggest the author looks at the submerged cylinder problem, near to, and both above and below, the interfacial layer.
WU: I will investigate this problem.

MILOH: A typical ratio of $\Delta\rho/\rho$ in the ocean is 10^{-4} and it hardly exceeds $3\ 10^{-2}$ (lakes or fjords). For these values it is very unlikely that there will be any effect on the added mass and damping coefficient as compared to the homogeneous case. The only pronounced effect of such a stratification is on the wave drag at the vicinity of the critical densimeter Froude number.
WU: For a floating vertical circular cylinder in two layer fluids of different densities, our numerical results had indicated that, if the body is in the upper layer water and very near the interface of the two fluids, the effects of the density difference on the added and damping coefficients may be remarkable as compared to the homogeneous case, in spite of the density being small.