# PERTURBATION RESULTS FOR THE RESONANCES OF THE SEA-KEEPING PROBLEM

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## Introduction:

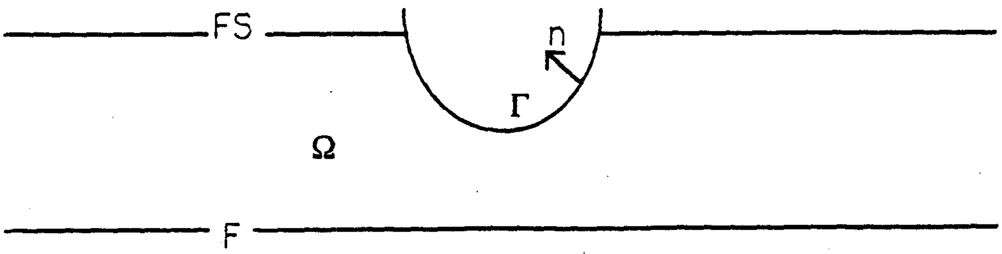
We are concerned with the motions of a floating body subjected to an incident wave. We study the variations of the amplitude of the motion with respect to the frequency of the excitation. The frequencies for which this amplitude is maximum are called "resonances of the problem". Our approach is to consider the resonances as the trace of complex singularities for the extension of the problem to complex frequencies. The singularities are called "scattering frequencies". A resonance lies near the real part of a scattering frequency whose imaginary part is small.

We limit our study to the first order linearization of the sea-keeping problem. We consider the 2-dimensional sea-keeping problem with finite depth. C. Hazard [1] has characterized the scattering values by means of a method developped by M. Lenoir and A. Tounsi [2]. They have stated that the problem is equivalent to one posed in a bounded domain by introducing an explicit sum at the boundary. The numerical method based on this idea is called the localized finite element method. C. Hazard [1] (part.III) has used this method to extend the problem to complex frequencies. He has computed the scattering frequences of this extended problem which are near the real axis and not too great (with respect to the absolute value). He has stated that the scattering values are solution of a nonlinear equation.

A basic difficulty is to find good starting values for the computation of the solution of this equation. We will introduce a small perturbation in the problem. We state the continuity of the scattering values with respect to the perturbation. We will continuously obtain the scattering values which come from the eigenvalues of the sea-keeping problem in a pool. This last problem is self-adjoint and classical programs for the calculation of eigenvalues of symmetric matrices are available.

## I) The classical sea-keeping problem

Let us consider a rigid body $(\mathcal{C})$ floating on the free surface. $(x, z)$ are the horizontal and vertical coordinates. The following notation characterized the system at rest: $\Omega$ is the fluid domain; $\Gamma$ is the hull of the body $(\mathcal{C})$; FS is the free surface ($\subset \{z = 0\}$); F is the bottom ($= \{z = -h\}$); $n$ is the unit outward normal on $\partial\Omega$.




We study "the linearized steady-state problem", i.e., the periodic motion of the body subjected to a sinusoidal incident wave of frequency $\omega$. We want to determine the scattered potential $\Phi$ as well as the motions of the body $(\mathcal{C})$, characterized by the three components of a vector $s = (s_1, s_2, s_3)$ (2 translations, one rotation).

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The couple $(\Phi, s)$ is solution of the following problem $Q_\nu$(where $\nu = \omega^2$) associated with the data $(g^{(1)}, g^{(2)})$ which characterized the external forces (they depend on the considered incident wave).

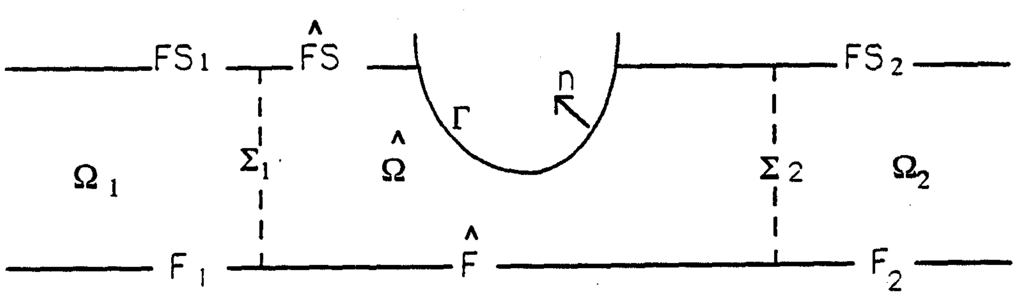
$$Q_\nu \quad \begin{cases} (1) & \Delta\Phi = 0 & \text{in } \Omega \\ (2) & \partial_n\Phi - \nu\Phi = 0 & \text{on FS} \\ (3) & \partial_n\Phi = 0 & \text{on F} \\ (4) & \partial_n\Phi + \sqrt{\nu}\, s\mathcal{N} = g^{(1)} & \text{on } \Gamma \\ (5) & (-\nu\mathbb{M} + \mathbb{K})s + \sqrt{\nu}\int_\Gamma \Phi\mathcal{N}\, d\gamma = g^{(2)} & \\ (6) & \int_{-h}^{0} \left|\frac{\partial\Phi}{\partial x} \mp i\nu_0\Phi\right|^2 dz \xrightarrow[x \to \pm\infty]{} 0 & \text{(radiation condition)} \\ & \nu_0 \text{ positive solution of} & \nu_0\tanh(\nu_0 h) = \nu \end{cases}$$

$\mathbb{M}$ is the 3×3 mass matrix of the body. $\mathbb{K}$ is the 3×3 hydrostatic stiffness matrix. $\mathcal{N}$ is the generalized outward normal on $\Gamma$ ( $\mathcal{N}(M) = (\underline{n}, \overline{OM} \wedge \underline{n})$ ).

To extend the problem to complex frequencies, we follow C. Hazard's arguments [1] based on the formulation of an equivalent problem posed in a bounded domain.

## II) The problem posed in a bounded domain.

Let $\Sigma_1, \Sigma_2$ be two vertical segments strictly enclosing the body $(C)$. We denote by $\hat{\Omega}$, $\Omega_1$ and $\Omega_2$ the inside, left and right fluid domain. $\hat{FS}$, $FS_1$, $FS_2$ and $\hat{F}$, $F_1$, $F_2$ are respectively the inside, left and right free surfaces and bottoms.




We define for $l = 1,2$, the boundary operator $C_{l,\nu}$ which maps a function $\chi$ defined on $\Sigma_l$ onto the normal derivative of the outer solution $\check{\Phi}$ of the outer Dirichlet problem:

$$\check{Q}_{l,\nu}(\chi) \quad \begin{cases} (1) \text{ in } \Omega_l \;\; (2) \text{ on } FS_l \;\; (3) \text{ on } F_l \\ (5)' \;\; \check{\Phi} = \chi \;\; \text{on } \Sigma_l \\ (6) \text{ the radiation condition} \end{cases}$$

The operator $C_{l,\nu}$ is explicitly known (see C. Hazard [1] part. III); the method of separation of variables provides the explicit sum:

$$C_{l,\nu}(\chi) = -i\nu_0(\nu)\left(\int_{\Sigma_l} \chi\Phi_\nu^{(0)}\, dz\right) \Phi_\nu^{(0)}(z) + \sum_{m=1}^{\infty} \zeta_\nu^{(m)} \left(\int_{\Sigma_l} \chi\Phi_\nu^{(m)}\, dz\right) \Phi_\nu^{(m)}(z)$$

$\zeta_\nu^{(m)}$ are the zeros of the dispersion equation: $\zeta\tanh(\zeta h) = -\nu$.

$\Phi_\nu^{(0)}(z) = a_\nu^{(0)}\cosh(\nu_0(z+h))$ corresponds to a radiative solution of $\check{Q}_{l,\nu}$. $\Phi_\nu^{(m)}(z)$ are terms which correspond to evanescent solutions of $\check{Q}_{l,\nu}$.

We now define the inside problem posed in the bounded domain $\hat{\Omega}$ :

Find $(\hat{\Phi}, \hat{s})$ such that $\hat{Q}_\nu$ holds,

$$\hat{Q}_\nu \quad \begin{cases} (1) \text{ in } \hat{\Omega} \quad (2) \text{ on } \hat{F}S \quad (3) \text{ on } \hat{F} \\ (4) \text{ on } \Gamma \quad (5) \text{ in } C^3 \\ (6)' \ \partial_n \hat{\Phi} = -C_{l,\nu}(\hat{\Phi}) \text{ on } \Sigma_l \ l = 1,2 \end{cases}$$

We show that $\hat{Q}_\nu$ is equivalent to the linearized steady-state problem $Q_\nu$. The extension of the problem to complex $\nu$ can be done through the explicit continuation of the boundary operators $C_{l,\nu}$ $l = 1,2$ for $\nu \in C$ (except perhaps the $\nu$ for which the dispersion equation $\zeta \tanh(\zeta h) = -\nu$ has a double root). From the variational formulation of the problem $\hat{Q}_\nu$ for $\nu \in C$, it follows that

$$(0) \qquad (\hat{S}(\nu) + \hat{T}_0(\nu) + \hat{T}_1(\nu))(\hat{\Phi}, \hat{s}) = \mathcal{F}(\nu)$$

where $\mathcal{F}(\nu)$ represents the external forces in $H^1(\hat{\Omega}) \times C^3$ and is the data, $\hat{T}_0(\nu)$ ( $\hat{T}_1(\nu)$ ) is the linear operators in $H^1(\hat{\Omega}) \times C^3$ associated with the radiative boundary term (resp. the evanescent boundary ones). $\hat{S}(\nu)$ stands for all the other terms.

Problem $\hat{Q}_\nu$ is well posed for all $\nu$ such that $Im(\nu) > 0$. The operator $\hat{R}(\nu) = (\hat{S}(\nu) + \hat{T}_0(\nu) + \hat{T}_1(\nu))^{-1}$ which maps the data $\mathcal{F}(\nu)$ onto the solution of $\hat{Q}_\nu$ is well defined for $\nu$ such that $Im(\nu) > 0$. It is analytic and has a meromorphic continuation in $\{\nu \in C \; ; \; Im(\nu) \leq 0 \}$. The poles are the $\nu$ for which the homogeneous equation (0) admits a non zero solution:

$$(1) \qquad ( \hat{S}(\nu) + \hat{T}_0(\nu) + \hat{T}_1(\nu) )(\hat{\Phi}, \hat{s}) = 0$$

In order to calculate them, we solve $\lambda(\nu) = 0$ where $\lambda(\nu)$ is an eigenvalue of $(\hat{S}(\nu) + \hat{T}_0(\nu) + \hat{T}_1(\nu))$. We need an iterative method (fixed point method or Newton's one). Finding good starting values for the computation is one difficulty of the problem. In order to find some, we introduce a small perturbation in the problem $\hat{Q}_\nu$ so that the scattering values are close to seeked ones.

## III) The $\varepsilon$ perturbated problem

Let us formulate the inside problem $\hat{Q}_\nu^\varepsilon$ by the same equations as $\hat{Q}_\nu$ except $(6)'$:
Find $(\hat{\Phi}^\varepsilon, \hat{s}^\varepsilon)$ such that $\hat{Q}_\nu^\varepsilon$ hods,

$$\hat{Q}_\nu^\varepsilon \quad \begin{cases} (1) \text{ to } (5) \\ (6)'' \ \partial_n \hat{\Phi}^\varepsilon = \varepsilon . C_{l,\nu}(\hat{\Phi}^\varepsilon_{|\Sigma_l}) \text{ on } \Sigma_l \text{ for } l = 1,2 \end{cases}$$

Notice that, for $\varepsilon = 1$, $\hat{Q}_\nu^1 = \hat{Q}_\nu$ is the sea-keeping problem in open sea and that, for $\varepsilon = 0$, $\hat{Q}_\nu^0$ is the sea-keeping problem in a pool.

The equation associated with $\hat{Q}_\nu^\varepsilon$ by means of the variational formulation is ( $\hat{S}(\nu) + \epsilon(\hat{T}_0(\nu) + \hat{T}_1(\nu))$ ) $(\hat{\Phi}^\varepsilon, \hat{s}^\varepsilon) = \mathcal{F}(\nu)$. We define, as for the unperturbed operator, the associated operator $\hat{R}^\varepsilon(\nu) = ( \hat{S}(\nu) + \varepsilon(\hat{T}_0(\nu) + \hat{T}_1(\nu)) )^{-1}$which maps the data $\mathcal{F}(\nu)$ onto the solution of $\hat{Q}_\nu^\varepsilon$.

The scattering values of $\hat{Q}_\nu^\varepsilon$ are the discrete poles of $\hat{R}^\varepsilon(\nu)$ in $\{z \in C \; / \; Im(z) \leq 0 \}$. They are the values of $\nu$ for wich the homogeneous problem associated with $\hat{Q}_\nu^\varepsilon$ admits a non zero solution:

$$(2) \qquad ( \hat{S}(\nu) + \epsilon(\hat{T}_0(\nu) + \hat{T}_1(\nu)) )(\hat{\Phi}^\varepsilon, \hat{s}^\varepsilon) = 0$$

## IV) Properties of the scattering values

<u>Proposition1</u>: *Scattering values are continuous with respect to $\varepsilon$. If $\nu_0$ is a scattering value of $\hat{Q}_\nu^{\varepsilon_0}$, then, there is a scattering value $\nu(\varepsilon)$ of $\hat{Q}_\nu^\varepsilon$ lying in the vicinity of $\nu_0$ and there is a $p \in \mathbb{N}^*$ such that $\nu(\varepsilon)$ is analytic with respect to $(\varepsilon - \varepsilon_0)^{\frac{1}{p}}$ if $\varepsilon$ lies in the vicinity of $\epsilon_0$.*

Remarks: The main tool is Steinberg's perturbation Theorem.4 (see [4]). A numerical method follows: first, calculate the eigenvalues of the rigid body ($C$) floating in the pool $\hat{\Omega}$; next use these values as the starting values for the calculation of the scattering values of $\hat{\mathbf{Q}}_\nu^\varepsilon$ for small $\varepsilon$; then with these new values , we can continue the process and use them to find the scattering values for greater $\varepsilon$; and so on until we attain $\varepsilon = 1$.

Proposition2: *Let $\nu_0$ be a simple root of the problem* (2) *for $\varepsilon_0 = 0$; then there is a unique pole $\nu(\varepsilon)$ in the vicinity of $\nu_0$, analytic with respect to $\varepsilon$ at $\varepsilon = 0$. Moreover,*

$$Re(\frac{d\nu}{d\varepsilon}(0)) = \frac{< \hat{T}_1(\nu_0)(\hat{\Phi}_0, \hat{s}); (\hat{\Phi}_0, \hat{s}_0) >}{- < \frac{\partial \hat{S}}{\partial \nu}(\nu_0)(\hat{\Phi}_0, \hat{s}_0);(\hat{\Phi}_0, \hat{s}_0) >}$$

$$Im(\frac{d\nu}{d\varepsilon}(0)) = \frac{< \hat{T}_0(\nu_0)(\hat{\Phi}_0, \hat{s}); (\hat{\Phi}_0, \hat{s}_0) >}{- < \frac{\partial \hat{S}}{\partial \nu}(\nu_0)(\hat{\Phi}_0, \hat{s}_0);(\hat{\Phi}_0, \hat{s}_0) >}$$

Remarks: $\nu_0$ is a simple root of (2) means that for $\varepsilon$ near 0 and $\nu$ near $\nu_0$, there is only a one dimensional algebraic eigenspace associated with a single eigenvalue of $\hat{S}(\nu) + \varepsilon(\hat{T}_0(\nu) + \hat{T}_1(\nu))$ near 0. $(\hat{\Phi}_0, \hat{s}_0)$ is the unique non zero vector solution of (2) for $\varepsilon = 0$ and $\nu = \nu_0$. The proof is based on the same arguments as M. Vullierme-Ledard [5]. The implicit function theorem gives the two above equalities.

We have followed four scattering values of $\hat{\mathbf{Q}}_\nu^\varepsilon$ in the complex plane for $\varepsilon \in [0,1]$. We have noticed that the derivatives at $\varepsilon = 0$ are nearly purely imaginar. The above equations give that the radiative term $\hat{T}_0$ brings the major perturbation. It may occur that two scattering values intersect. If the intersection occurs for the same $\varepsilon$, i.e., $\nu_1(\varepsilon) = \nu_2(\varepsilon) = \nu$, $\nu$ is at least of order two. If not, $\nu_1(\varepsilon_1) = \nu_2(\varepsilon_2) = \nu$ implies that $\nu$ is a scattering value for two different problem $\hat{\mathbf{Q}}_\nu^{\varepsilon_1}$ and $\hat{\mathbf{Q}}_\nu^{\varepsilon_2}$.

## Conclusion

We state also the continuity with respect to the geometry of the hull or a wall that encloses the body. But the theory of perturbation used by S. Steinberg [4] is not usable when the wall attains the free surface. Techniques as J. T. Beale's [3] in the case of resonators in acoustics provide the same results in hydrodynamics.

References:


[1] C. Hazard, *Etudes des résonances pour le problème linéarisé des mouvements d'un navire sur la houle*, Thèse de l' Université PARIS VI, (1991).

[2] M. Lenoir and A. Tounsi, *The localised finite element method and its application to the two dimensional sea-keeping problem*, S.I.A.M. Journal of numerical analysis, vol. 25 n°4, (1988).

[3] J. T. Beale, *Scattering Frequencies of Resonators*, Communication on Pure and Applied Mathematics, vol. XXVI, p.549-563, (1973).

[4] S. Steinberg, *Meromorphic families of operators*, Arch. Ration. of Mechanical Analysis, vol. 31, p.372-380, (1968).

[5] M. Vullierme-Ledard, *Asymptotic study of the vibration problem for an elastic body deeply immersed in an incompressible fluid*, Mathematical Modelling and Numerical Analysis, vol. 19, n°1, p.145-170, (1985).

[6] C. Hazard, M. Lenoir, *Numerical computation of resonant states for the 2-D linear sea-keeping problem*, Fifth international workshop on water-waves and floating bodies, Manchester, England, (1990).

## DISCUSSION

MARTIN: Two questions on your numerical results, showing the movement of the scattering frequencies as $\varepsilon$ is varied:

What is the geometry of the floating bodies?

Can you prove that the real frequencies at $\varepsilon = 0$ move off the real axis perpendicularly for small $\varepsilon$?

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Answer to question 1: The body consists of 2 rigid hulls linked by an elastic beam which is lying above the water: it is a catamaran.

Answer to question 2: The real scattering values at $\varepsilon = 0$ don't move off perpendicularly but nearly perpendicularly. As explained in the notes, this means that if we consider the eigenfunction associated to one of these V(0), its boundary components along radiative terms are greater than the components along evanescent ones.

We can explicitly calculate those eigenfunctions for a pool (without body!) and verify that. For a general body, I can't see why this occurs.

Remark: In your work with Luke, you obtained similar results. In your case, for infinite depth ($\varepsilon = 0$), the components of eigenfunctions along radiative terms are zero so the scattering values move nearly along the real axis (and even exponentially as you prove it.)