# Interaction Of Water Waves With Thin Plates

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## 1 Introduction

Consider the scattering of two-dimensional surface water waves by a submerged thin plate, $\Gamma$; we assume that $\Gamma$ is a curved arc of finite length. Historically, most of the work done on this scattering problem assumes that the plate is flat and horizontal or vertical, and that the water is deep. Evans [1] has given an exact solution for a vertical plate, and reviews earlier work. Horizontal plates in water of finite depth have been considered by Patarapanich [7] (using finite elements) and by McIver [5] (using matched eigenfunctions). These two methods cannot readily treat deep water and/or angled plates, and they cannot readily use information on the known singular behaviour near the plate edges. In addition, it is difficult to impose the radiation condition with the finite-element method.

Here, the problem is reduced to the solution of a hypersingular integral equation for the discontinuity in potential across the plate. Once found, this discontinuity can be used for the direct calculation of the reflection and transmission coefficients, $\mathcal{R}$ and $\mathcal{T}$. We show how an approximate solution can be obtained by using a truncated series of Chebyshev polynomials of the second kind, multiplied by an appropriate weight function. The unknown coefficients in such an expansion can be found by collocation, and, once found, can be used for the direct calculation of $\mathcal{R}$ and $\mathcal{T}$. We have used this method for a flat plate, submerged in deep water. We have validated the method by reproducing published graphs of $|\mathcal{R}|$ for vertical and horizontal plates. We also give some new results for an inclined plate that makes an angle of $\pi/4$ to the vertical.

## 2 Formulation

Classical linear water wave theory is used and all motion is assumed to be simple harmonic in time with angular frequency $\omega$. With these assumptions, we can introduce the harmonic velocity potential $\Re\{\phi(x,y)e^{-i\omega t}\}$. An incident train of waves with potential $\phi_{\text{inc}}$ is assumed, whereby the total potential may be written as $\phi = \phi_{\text{inc}} + \phi_{\text{sc}}$, where the scattering potential $\phi_{\text{sc}}$ is sought. As well as being harmonic, $\phi_{\text{sc}}$ must satisfy

$$K\phi_{\text{sc}} + \frac{\partial \phi_{\text{sc}}}{\partial y} = 0 \qquad \text{on the free surface, } y = 0, \tag{1}$$

and

$$\frac{\partial \phi_{\text{sc}}}{\partial n_q} = -\frac{\partial \phi_{\text{inc}}}{\partial n_q} \qquad \text{on the plate, } \Gamma. \tag{2}$$

Here, $K = \omega^2/g$, $g$ is the acceleration due to gravity and $\partial/\partial n_q$ denotes normal differentiation at the point $q$ on $\Gamma$. Choosing the appropriate fundamental solution $G$ and applying Green's theorem to $\phi_{\text{sc}}$ and $G$, we find

$$\phi_{\text{sc}}(P) = \frac{1}{2\pi} \int_\Gamma [\phi(q)] \frac{\partial G(P,q)}{\partial n_q}\, ds_q, \tag{3}$$

where $P$ is any point in the fluid. Applying (2) to (3), we find [6]

$$\frac{1}{2\pi} \,⨎_\Gamma [\phi(q)] \frac{\partial^2 G(p,q)}{\partial n_p \partial n_q}\, ds_q = -\frac{\partial \phi_{\text{inc}}}{\partial n_p}, \qquad p \in \Gamma, \tag{4}$$

which is to be solved for $[\phi(q)]$; the integral is to be interpreted as a Hadamard finite-part integral.

## 3 Method of solution

Consider a flat plate of length $2a$; we take $a = 1$, without loss of generality. So for a flat plate in deep water, parametrisation of (4) leads to

$$\oint_{-1}^{1} \frac{f(t)}{(s-t)^2}\, dt + \int_{-1}^{1} f(t) L(s,t)\, dt = 2\pi h(s), \qquad -1 < s < 1, \tag{5}$$

where

$$L(s,t) = \frac{Y^2 - X^2}{(X^2+Y^2)^2} + \frac{2KY}{X^2+Y^2} + 2K^2 \oint_0^\infty e^{-kY} \cos kX \frac{dk}{k-K}, \tag{6}$$

$h(s) = \partial\phi_{\text{inc}}/\partial n_q$, $X = (s-t)\sin\alpha$, $Y = (s+t)\cos\alpha + 2d$ and $f(t) = [\phi(q)]$. Also, $d$ is the submergence of the mid-point of the plate and $\alpha$ is the angle that the plate makes with the vertical. For a submerged plate, the leading behaviour of $f(t)$ at the edges of the plate is known to be [6]

$$f(t) \sim \sqrt{1 \mp t}\, f_{\mp} \qquad \text{as } t \to \pm 1, \tag{7}$$

where $f_{\mp}$ are constants. We build this into a numerical procedure for solving (5) by writing

$$f(t) = \sqrt{1-t^2}\, g(t), \tag{8}$$

and then approximate $g(t)$ using a polynomial. Since

$$\oint_{-1}^{1} \frac{\sqrt{1-t^2}\, U_n(t)}{(s-t)^2}\, dt = -\pi(n+1) U_n(s), \tag{9}$$

where $U_n(t)$ is a Chebyshev polynomial of the second kind, a natural choice is

$$g(t) \cong \sum_{n=0}^{N} a_n U_n(t) \equiv g_N(t), \tag{10}$$

say; by definition,

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}. \tag{11}$$

We determine the unknown coefficients $a_n$ by a straightforward collocation scheme,

$$\sum_{n=0}^{N} a_n \left[-\pi(n+1) U_n(s_j) + K_n(s_j)\right] = 2\pi h(s_j), \qquad j = 0, 1, \ldots, N, \tag{12}$$

where

$$K_n(s_j) = \int_{-1}^{1} U_n(t) \sqrt{1-t^2}\, L(s_j, t)\, dt \tag{13}$$

and the collocation points $s_j$ are chosen as

$$s_j = \cos\left(\frac{2j+1}{N+1}\frac{\pi}{2}\right), \qquad j = 0, 1, \ldots, N.$$

Golberg [2] has proved that this method is uniformly convergent, so that

$$\max_{-1 \le t \le 1} |g(t) - g_N(t)| \to 0 \qquad \text{as } N \to \infty.$$

## 4 Results

The method has been validated against published solutions for vertical and horizontal plates. As an example of its efficacy for another configuration, we have computed the reflection and transmission coefficients for the two-dimensional scattering of water waves by a flat, submerged plate at an angle of $\pi/4$ to the vertical. It is straightforward to show that, in terms of the unknown coefficients in the expansion (10), the reflection coefficient for any $\alpha$ is given by

$$\mathcal{R} = -Ke^{-Kd-i\alpha} \sum_{n=0}^{N} a_n \left[\overline{L}_n - \overline{L}_{n+2}\right] \tag{14}$$

with the corresponding transmission coefficient given by

$$\mathcal{T} - 1 = Ke^{-Kd+i\alpha} \sum_{n=0}^{N} a_n \left[L_n - L_{n+2}\right], \tag{15}$$

where $L_n = (\pi/2)(-1)^n I_n(Ke^{i\alpha})$ and $I_n$ is a modified Bessel function. The overbar denotes the complex conjugate. Figure 1 contains graphs of $|\mathcal{R}|$. The value of $N$ was 15 for these results.

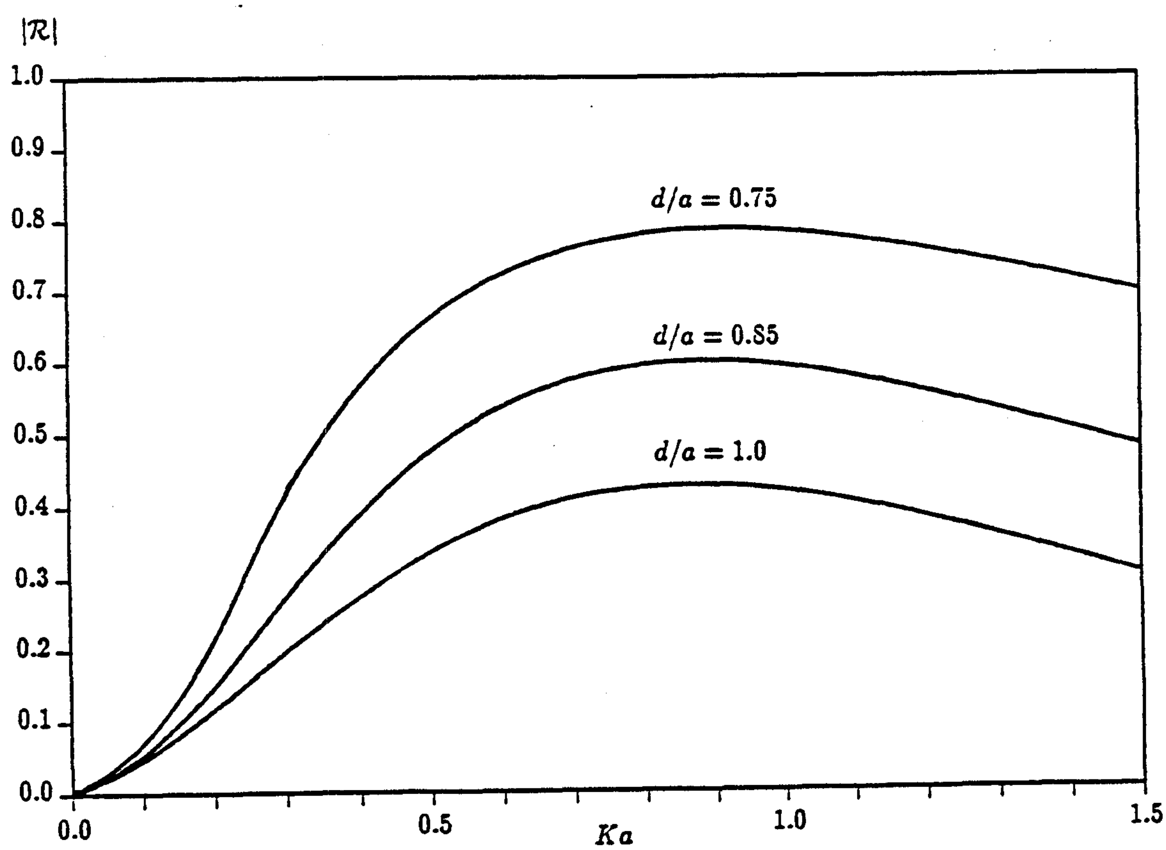


Figure 1: Reflection coefficients for a plate at an angle of $\pi/4$ to the vertical, plotted against $Ka$ for various values of $d/a$.

# 5 Discussion

We are currently developing the above method in two directions: surface-piercing plates and trapped modes.

## 5.1 Surface-piercing plates

For a surface-piercing plate, it is convenient to use a different parametrisation of the plate, so that $t = 1$ corresponds to the lower edge and $t = 0$ corresponds to the point where the plate meets the free surface. The behaviour of $f(t)$ near $t = 0$ is no longer a square-root zero: $f(0)$ is an unknown finite constant. Nevertheless, we find that a similar method (Chebyshev polynomials and collocation) works well; the main difficulty is in controlling the strong singularities in $L$ at $X = Y = 0$.

We note that this problem has a long history, going back to Ursell's solution [8] for a vertical plate, and John's solution [3] for an inclined plate.

## 5.2 Trapped modes

Recently, there has been considerable interest in trapped modes. In particular, Linton & Evans [4] have computed trapped modes above a submerged, horizontal flat plate in water of finite depth, using matched eigenfunctions. The present method should extend to problems of this type, allowing computations for deep water and plates of any shape.

# References


[1] Evans, D.V. Diffraction of water-waves by a submerged vertical plate. *J. Fluid Mech.* **40** (1970) 433–451.

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## DISCUSSION

TUCK: The problem with $\alpha < 0$ ($|\alpha|$ small) would seem to have some relationship to a *beach* problem.

PARSON & MARTIN: Changing the sign of $\alpha$ is equivalent to changing the direction of the incident wave. It is well known that $\mathcal{T}$ and $|\mathcal{R}|$ are the same for both directions. However, we might not expect to see this equality in experiments, especially for surface-piercing plates, because wave breaking could occur for $\alpha < 0$, leading to a reduction in transmitted energy.