On Uniqueness and Asymptotic Behavior of Solutions of the Neumann–Kelvin Problem

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In the first part of the paper asymptotic representations at infinity of the Green function, of the velocity potential and of the elevation of the free surface are obtained for the linear problem on the uniform motion of a pointwise source in a fluid. The advantage of these representations is that they are uniform with respect to all directions and depth.

In the second part it is proved that the problem on the uniform motion of a submerged body is uniquely solvable for all velocities except, possibly, for a finite number of them.

The Green function of the Neumann–Kelvin problem satisfies the relations

$$\Delta G = -4\pi\,\delta(x, y, z - z_0)\,, \quad z < 0\,, \quad z_0 < 0$$

$$G''_{xx} + G'_z = 0\,, \quad z = 0$$

and it is defined by well known integral representations ([1]). The Kelvin velocity potential Φ corresponding to the moving pointwise source of intensity P and the elevation of the free surface η are expressed by the Green function G in the following way

$$\Phi = \frac{\partial}{\partial x}\left[\frac{P}{V\rho}\,G(\nu x, \nu y, \nu z, \nu z_0)\right], \quad \eta = \frac{\partial^2}{\partial x^2}\left[\frac{P}{g\rho}\,G(\nu x, \nu y, 0, \nu z_0)\right].$$

Here V is the speed of the source, g is the acceleration due to gravity, ρ is the density of the fluid, $\nu = g/V^2$.

Asymptotic behavior at infinity of the functions G, Φ, η was considered by many authors. In particular F. Ursell [3], [4], obtained asymptotic behavior of these functions when both the observation point and the source are at the free surface, that is $z = z_0 = 0$. It is noted in [3], [4], that this asymptotic behavior is non–uniform with the amplitude of one of the waves tending to infinity when the observation point is tending to the track of the source along the free surface. And the amplitude of this wave is vanishing when the observation point is under the track of the source (see also [2]).

We single out an additional wave concentrated along the track of the source and find asymptotic expansion in which the remainder decays at infinity uniformly with respect to all variables, in particular, in a neighborhood of the track of the source. Because of the lack of the space we shall give here only the asymptotic behavior of ∇G (in particular, it defines asymptotic behavior of Φ), and we shall give it strictly within the Kelvin angle only. Let

$$R_0 = \sqrt{x^2 + y^2 + (z - z_0)^2}\,, \quad R = \sqrt{x^2 + y^2 + (z + z_0)^2}\,,$$

r, ϕ be the polar coordinates on free surface, $0 \le \phi < 2\pi$,

$$\begin{aligned} \phi_0 &= \operatorname{arc} tg\,\sqrt{2/4} (= \operatorname{arc}\sin 1/3)\,, \\ t_\pm(\phi) &= -\frac{1}{4}(ctg\,\phi \pm ctg\,\phi\,\sqrt{1 - 8tg^2\,\phi})\,, \\ S_\pm(\phi) &= \sqrt{t_\pm^2(\phi) + 1}\,(\cos\phi + t_\pm(\phi)\,\sin\phi)\,, \\ a_\pm(\phi) &= -4\sqrt{2\pi}\,\left[\frac{t_\pm^2(\phi) + 1}{1 - 9\,\sin^2\phi}\right]^{1/4}. \end{aligned}$$

Theorem. *If* $|\phi - \pi| < \phi_0 - \epsilon$, $\epsilon > 0$, *then*

$$\begin{aligned} \nabla G = \nabla\,\frac{1}{R_0} &+ \operatorname{Im}\sum_\pm [H_\pm a_\pm(\phi)\,r^{-1/2}\,e^{(z+z_0)(t_\pm^2(\phi)+1)+iS_\pm(\phi)\,r \pm i\frac{\pi}{4}}] + \\ &+ \frac{\sqrt{\pi}}{|y|^{3/2}}\operatorname{Im}(H\,e^{z+z_0+iS_+(\phi)\,r+i\frac{\pi}{4}}) + O(R^{-3/2})\,, \end{aligned} \tag{1}$$

where the vectors $H_\pm$ *and* H *are*

$$\begin{aligned} H_\pm &= (i\,\sqrt{t_\pm^2(\phi)+1}\,,\ it_\pm(\phi)\,\sqrt{t_\pm^2(\phi)+1}\,,\ t_\pm^2(\phi)+1)\,, \\ H &= \left(\frac{2\,|y|}{r}\,D_1\,,\ (\operatorname{sign} y)\,D_2 + \frac{iy}{2r^2}\,D_3\,,\ -iD_2 + \frac{|y|}{2r^2}\,D_3\right) \end{aligned}$$

and

$$|O(R^{-3/2}| \le CR^{-3/2}\,,\ R \ge 1$$

with constant C *which does not depend on* x, y, z, z_0. *Here* $D_i = D_i((z + z_0)t_+^2(\phi))$ *and*

$$D_s(\xi) = \frac{d^2}{d\sigma^2}(\sigma^s\,e^{\xi\sigma^2})\big|_{\sigma=1}\,,\ s = 1, 2,\ D_3(\xi) = \frac{d^4}{d\sigma^4}(\sigma^2\,e^{\xi\sigma^2})\big|_{\sigma=1}\,.$$

Inside the Kelvin angle ($|\phi - \pi| < \phi_0$) the functions t_-, S_-, a_- are infinitely differentiable and the functions t_+, S_+, a_+ have singularities at $\phi = \pi$. So the terms of the right–side of (1), containing the functions t_+, S_+ have singularities at the track of the source ($\phi = \pi$, $z = z_0 = 0$). But the remainder is continuous and decreases uniformly. The last term of the right–side of (1) standing before the remainder decays rapidly at infinity if $|y| > \epsilon$ or $z + z_0 < -\epsilon$. This term describes a wave going along the track of the source.

Asymptotic behavior of the function G''_{xx}, which defines η, differs from asymptotic behavior of the function G'_z only by the sign. Similar assertions for functions $G, \nabla G, \Phi, \eta$ are obtained not only for the interior of the Kelvin angle but also for exterior of the angle and for a neighborhood of its boundary.

Now let B be a compact (a body) with a smooth boundary ∂B, placed in the half–space $\mathbb{R}^3_- = \{(x, y, z) : z < 0\}$. We look for a solution u of the problem

$$\begin{aligned} &\Delta u = 0\,, \quad (x,y,z) \in \mathbb{R}^3_-\backslash B; \\ &\frac{\partial^2 u}{\partial x^2} + \nu\,\frac{\partial u}{\partial z} = 0\,, \quad z = 0\,; \quad \frac{\partial u}{\partial n} = -V\,\cos(n,x) \quad \text{on} \quad \partial B\,, \end{aligned} \tag{2}$$

which can be represented as the potential

$$u = \int_{\partial B} G_\nu(x - x_0, y - y_0, z, z_0)\,\phi(x_0, y_0, z_0)\,dS_0\,, \tag{3}$$

where dS_0 is the surface element on ∂B, ϕ is some continuous function on ∂B,

$$G_\nu(x, y, z, z_0) = \nu G(\nu x, \nu y, \nu z, \nu z_0)$$

and G is the Green function of the Neumann–Kelvin problem.

Theorem. *The problem* (2), (3), *has the unique solution for all* $\nu > 0$ *except, possibly, a finite number of them.*

Concerning the last theorem we should like to recall our old results [5], [6], which give the sufficient conditions for the unique solvability of similar problems for all ν. The three–dimensional problem on steady–state oscillations of a layer of fluid of variable depth is uniquely solvable for all frequencies if the bottom is flat in a neighborhood of infinity and satisfies one of the two conditions:

1. The intersection of the domain occupied by the fluid with the plane $z = -H$ is starlike with respect to the point $(0, 0, -H)$ for any $H > 0$.

2. There is a point at the depth H, $H\nu \leq 1$, from which we can "see" all the bottom.

The homogeneous two–dimensional Neumann–Kelvin problem for submerged body has only trivial solution (for all ν) in the class of functions with bounded energy if the body is starlike with respect to x–axis.

References

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Discussion

Noblesse Can the asymptotic approximation you obtained for a point source be easily extended to the case of a continuous distribution of sources over a surface?

Vainberg Yes, it is very simple, because we must integrate only the main term of the asymptotic expansion since we have the uniform estimate of the remainder. A problem describing the motion of a ship on an air cushion is an example where we must consider the sources distributed over some domain of the free surface. With the help of the results given above one can easily find the asymptotic behavior of the solution of this problem at infinity. The singularities on the track will depend on the smoothness of the distribution of the sources.

If the solution of the problem on the uniform motion of a immersed body has the form of a potential whose density is distributed over the immersed part of the hull and the waterline, then it is also easy to find the asymptotic behavior of the solution at infinity.

Evans The uniqueness of the solution of the problem on steady–state oscillations requires the domain to be starlike so that from some point in the fluid one can "see" all other points. Is this restriction just a result of the method of the proof or is it possible that a non–starlike domain exists for which the solution of the problem is not unique, say in 2 dimensions? This could be some local oscillation of the fluid which does not radiate its energy to infinity. Is this likely?

Vainberg It is difficult to give a definite answer. On one hand the condition of starlikeness originates from the method of the proof. The natural operators connected with the problem are not positive. But the starlike nature of the domain leads to positivity of some quadratic form of the solution which gives us uniqueness of the solution. It is possible to find other conditions which would play the same role. But I don't know whether any conditions are necessary at all, because there is no theorem on the unique solvability of the problem without any conditions on the geometry of the bottom and of the floating body and there are no examples of a three–dimensional problem (not for a channel) with non–unique solution.

Pawlowski Could you please comment if the theorem about the existence and uniqueness of the solution to the Neumann–Kelvin problem for a 3–dimensional body applies only to submerged bodies or does it also apply to bodies which intersect the free surface?

Vainberg This theorem applies only to fully submerged bodies. If the body intersects the free surface, then in order for the problem to be well–posed we must also pose some boundary conditions on the waterline. I think that nobody knows the proper mathematical formulation of the problem in this case. But it would be very intersting to find it.

Wehausen I think that it has not been sufficiently emphasized in the lecture that the authors have finally filled the gap left by N. E. Kochin when he treated this problem in the late 1930's and was able to prove uniqueness and existence only for sufficiently large values of ν. One hopes that the authors will be able to treat successfully the same problem when the body intersects the free surface. It would not surprise me if it should turn out that no solution exists in this case.