ASYMPTOTIC STUDY OF SCATTERING FREQUENCIES FOR A COUPLED SYSTEM

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1 Introduction

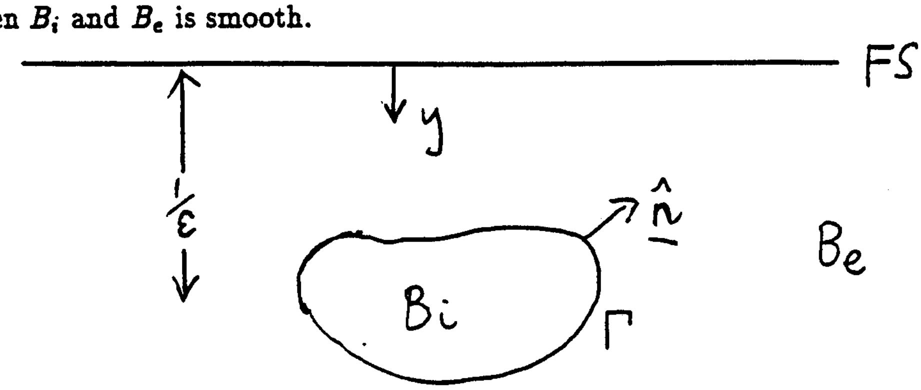
Consider an elastic body surrounded by an unbounded perfect fluid. Such a coupled system has a countable set of real scattering frequencies: at these frequencies, the governing equations have a non-trivial solution; in particular, the velocity potential in the fluid is not identically zero.

Suppose, now, that a free surface is introduced, so that the body is deeply submerged. What happens to the scattering frequencies? In her paper of 1985, Vullierme–Ledard showed that the scattering frequencies associated with simple modes have *purely real* asymptotic expansions in inverse powers of submergence depth. However, if the perturbed frequencies are indeed real, this would imply an unexpected non-uniqueness in the physical problem: we expect that the presence of the free suface will allow energy to escape.

Motivated by Vullierme–Ledard's work (see also her thesis, 1988), we study here a simpler problem involving an inclusion occupied by a compressible fluid instead of an elastic body. This problem is qualitatively similar to the former. We concentrate on the imaginary parts of the scattering frequencies and show that they are "exponentially small". Explicit results are given for the case of a spherical inclusion.

2 Formulation

The fluid occupies the domain $B_e \subset \mathcal{R}^3$ and the inclusion the domain B_i. The boundary Γ between B_i and B_e is smooth.



The linearised steady-state problem is studied; i. e. all time dependence is in the form $e^{-i\omega t}$. We have to find the velocity potentials Φ and Ψ such that

$$\nabla^2\Phi = 0 \qquad \text{in } B_e \tag{1}$$

$$(\nabla^2 + \omega^2)\Psi = 0 \quad \text{in } B_i \tag{2}$$

$$\Phi|_\Gamma = \Psi|_\Gamma \tag{3}$$

$$\frac{\partial \Phi}{\partial n}|_\Gamma = \frac{\partial \Psi}{\partial n}|_\Gamma \tag{4}$$

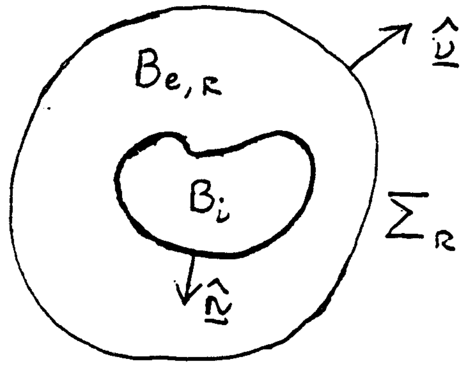
$$\left(\frac{\partial \Phi}{\partial y} + \omega^2 \Phi\right) |_{FS} = 0 \tag{5}$$

$$\lim_{\rho \to \infty} \int_0^\infty \int_0^{2\pi} \rho \mid \frac{\partial \Phi}{\partial \rho} - i\omega^2 \Phi \mid^2 d\theta dy = 0, \tag{6}$$

where (ρ, θ, y) denote cylindrical polar coordinates.

2.1 The truncated problem

We now proceed to study an equivalent problem in a bounded domain: the so-called truncated problem. Let $B_{e,R}$ be the domain delimited by the surfaces Σ_R and Γ, where Σ_R is the surface of a sphere of radius R, centred on $y = \frac{1}{\epsilon}$ and $\rho = 0$ and which completely contains B_i.



In the above formulation B_e is replaced by $B_{e,R}$ in equation (1). Equation (5) and the radiation condition (6) are replaced by

$$\frac{\partial \Phi}{\partial \nu} |_{\Sigma_R} = T(\omega^2; \epsilon)(\Phi |_{\Sigma_R}) \tag{7}$$

The operator $T(\omega^2; \epsilon)$ can be defined for non-real values of ω^2. It is the completion of a densely defined operator that can be explicitly written down. When ω^2 is real, it can be shown that

$$\left(\Im \int_{\Sigma_R} (T(\omega^2; \epsilon)\Phi)\overline{\Phi} dS\right) e^{\frac{2\omega^2}{\epsilon}} \leq O(1) \qquad \text{as } \epsilon \to 0$$

2.2 The exterior problem

We must find the potential Φ satisfying Laplace's equation in $B_{e,R}$, equation (7) and a Dirichelet boundary condition on the inner surface Γ. It can be shown that, if $\Phi |_\Gamma = g \in H^{\frac{1}{2}}(\Gamma)$, then a unique solution for $\Phi \in H^1(B_{e,R})$ exists. Moreover, the normal derivative of Φ on Γ can be defined and is a member of $H^{-\frac{1}{2}}(\Gamma)$. Let us define an operator $S(\omega^2; \epsilon) \in \mathcal{L}(H^{\frac{1}{2}}(\Gamma), H^{-\frac{1}{2}}(\Gamma))$ thus:

$$\frac{\partial \Phi}{\partial n} |_\Gamma = S(\omega^2; \epsilon) g$$

2.3 The interior problem

We now take for the datum g in the exterior problem the trace of some potential $\Psi \in H^1(B_1)$ that satisfies equation (2) and impose the extra constraint

$$\frac{\partial \Psi}{\partial n} |_\Gamma = S(\omega^2; \epsilon)(\Psi |_\Gamma).$$

By applying Green's theorem, we can see that

$$\omega^2 \int_{B_i} \Psi \overline{\Psi_1} dV = \int_{B_i} \nabla \Psi . \nabla \overline{\Psi_1} dV - \int_\Gamma (S(\omega^2; \epsilon)\Psi) \overline{\Psi_1} dS \tag{8}$$

is true for any $\Psi_1 \in H^1(B_i)$.

The right-hand side of equation (8) can be written as a duality product

$$< B(\omega^2; \epsilon)\Psi, \Psi_1 >_{H^{-1}(B_i), H^1(B_i)}$$

and the left-hand side of (8) can be thought of as the duality product between $\Psi_1 \in H^1(B_i)$ and $\Psi \in H^{-1}(B_i)$. As equation (8) is true for all $\Psi_1 \in H^1(B_i)$, the equation satisfied by Ψ is

$$B(\omega^2; \epsilon)\Psi = \omega^2 \Psi. \tag{9}$$

We can show that the operator $B(\omega^2; \epsilon)$ has a countable set of eigenvalues. In general, these will depend on ω^2 and, therefore, equation (9) defines an implicit eigenvalue problem.

The eigenvalues of $B(\omega^2; 0)$ are all real and are independent of ω^2. Furthermore, their associated eigenspaces are finite dimensional. It is clear, then, that the implicit eigenvalue problem is solvable when $\epsilon = 0$.

3 Results

If we consider only those eigenvalues whose associated eigenspaces have multiplicity one (simple eigenvalues), then it can be shown that these eigenvalues remain simple as ω^2 and ϵ are perturbed and that they depend analytically on ω^2.

If $\lambda(\omega^2; \epsilon)$ is a simple eigenvalue, then define

$$\Upsilon(\omega^2; \epsilon) = \lambda(\omega^2; \epsilon) - \omega^2.$$

Denote by $\omega^2(0)$ a scattering frequency of the unperturbed problem: that is to say,

$$\Upsilon(\omega^2(0); 0) = 0. \tag{10}$$

Clearly,

$$\frac{\partial \Upsilon}{\partial \omega^2} |_{\omega^2 = \omega^2(0), \epsilon = 0} = -1. \tag{11}$$

In addition to equations (10) and (11) we have

$$\Im \Upsilon(\omega^2, \epsilon) \sim A(\omega^2) e^{-\frac{2\omega^2}{\epsilon}} (1 + o(1)) \qquad \text{as } \epsilon \to 0, \tag{12}$$

for real ω^2 and where $A(\omega^2)$ is analytic in ω^2.

Equations (10)–(12) imply the existence of a locally unique function $\omega^2(\epsilon)$ which has an asymptotic expansion in integer powers of ϵ. The coeficients of the powers of ϵ in this expansion are all real.

Define $\chi(x;\epsilon) \equiv \Im\Upsilon(\Re\omega^2(\epsilon) + ix;\epsilon)$ and, for fixed ϵ, treat $\chi(x;\epsilon)$ as a real function of the real variable x in the range $[\Im\omega^2(\epsilon), 0]$, then by using the mean value theorem, we have

$$\chi(\Im\omega^2(\epsilon);\epsilon) - \chi(0;\epsilon) = \Im\omega^2(\epsilon)\frac{\partial\chi(x;\epsilon)}{\partial x}\Big|_{x=x_\epsilon}$$

at some point $x_\epsilon \in [\Im\omega^2(\epsilon), 0]$.

The left-hand side of the above equation is clearly equal to $-A(\Re\omega^2(\epsilon))e^{-\frac{2\Re\omega^2(\epsilon)}{\epsilon}}$.

$$\lim_{\epsilon\to 0}\frac{\partial\chi(x,\epsilon)}{\partial x}\Big|_{x=x_\epsilon} = \lim_{\epsilon\to 0}\Re\frac{\partial\Upsilon}{\partial\omega^2}\Big|_{\omega^2=\Re\omega^2(\epsilon)+ix_\epsilon} = -1$$

using equations (11) and (12).

Therefore, the leading order term in the asymptotic expansion of $\Im\omega^2(\epsilon)$ is

$$A(\omega^2(0))e^{-2b}e^{-\frac{2\omega^2(0)}{\epsilon}},$$

where b is the coefficient of the ϵ term in the asymptotic expansion of $\omega^2(\epsilon)$.

It can be shown that $A(\omega^2)$ is never positive when ω^2 is real. Thus, the scattering frequencies never move into the upper half-plane. This is consistent with the result that the only solution of the homogeneous problem is the trivial solution when $\Im\omega^2$ is greater than zero.

For the case of a spherical inclusion, the expressions for b and $A(\omega^2(0))$ can be written down explicitly. For a sphere of unit radius, the $(m,n)th$ interior potential of the unperturbed problem has the form $j_n(\omega^2(0)r)P_n^m(\cos\theta)(a\cos m\phi + b\sin m\phi)$, where j_n is the nth spherical Bessel function, (r,θ,ϕ) are spherical polar coordinates and $\omega^2(0)$ is a zero of j_{n-1}. $A(\omega^2(0)$ and b are

$$-\frac{8\pi(n+\frac{1}{2})(\omega^2(0))^{2n+\frac{5}{2}}j_n^2(\omega^2(0))}{(n+m)!(n-m)!\int_0^{\omega^2(0)}x^2j_n^2(x)dx}$$

$$b = \begin{cases} 0 & \text{if } n \neq 0 \\ \frac{\omega^3(0)j_0^2(\omega^2(0))}{\int_0^{\omega^2(0)}x^2j_0^2(x)dx} & \text{if } n = 0 \end{cases}$$

4 References

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DISCUSSION

URSELL: You might expect the damping to be exponentially small, since any disturbance at depth $1/\varepsilon$ gives rise to a wave amplitude proportional to $\exp(-k/\varepsilon)$.
LUKE & MARTIN: We agree. Indeed, it was this observation that led us to believe that Vullierme-Ledard's results were incomplete.

KUZNETSOV: To obtain your result, you use the usual function spaces and consider three problems in different domains. I want to bring to your attention the paper by Agmon & Hörmander [*J. Analyse Math.* **30** (1976) 1-38] (see also the chapter entitled 'Scattering Theory' in Hörmander's book). They have introduced a special function space that includes the radiation condition. Their technique may be useful for your purposes.
LUKE & MARTIN: Thank you for the references. However, we see no advantage in using the technique mentioned over that used by us in which the radiation condition is built into the operator instead of into the function space. Moreover, our technique has distinct advantages if one is interested in actually computing the scattering frequencies, as any integration is over a finite domain.