Numerical Solution of Nonlinear Wave-Bottom Interactions Using Numerical Conformal Mapping

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In the past few decades, the mixed Eulerian and Lagrangian method (MEL) based on Longuet-Higgins' work has become one of the most popular approaches to deal with transient nonlinear wave-body problems. The main idea behind this method is based on the time-stepping concept, such that an 'initial boundary value problem' is decomposed into a series of 'linear' boundary value problems which are then solved by the Boundary Element Method (BEM). However, one difficulty associated with this numerical scheme is caused by the spatial singularity which appeared at the intersection of the solid body with the free surface. Therefore, the treatment of singularities becomes one of the most important subjects in the application of BEM. From recent published literatures and our experiences, it seems that the BEM cannot generally provide an accurate solution in the vicinity of corners on a boundary, even if the solution is regular at these corners. Sometimes, the accuracy of the numerical solution can be improved a little bit if finer meshes is used, but this may cause numerical instability in the computation.

In order to resolve the difficulty mentioned above, an indirect numerical scheme developed from the generalized Schwarz-Christoffel Transformation is adopted in our work to solve the mixed Dirichlet-Neumann problem with corner singularities. Although the CPU time may be increased for solving a single boundary value problem because of the iterative nature of the numerical conformal mapping, the accuracy of the numerical solution in the vicinity of the corner is improved dramatically.

It is assumed that the viscous and compressible effects are negligiable and the flow is irrotational. The nonlinear wave-bottom interaction problem can be interpreted as the waves generated by a wavemaker on the left end, propagating towards the right over a bottom. If the problem is formulated in a moving coordinate system fixed on the wavemaker (see Fig.1), the velocity potential $\phi(x,y,t)$ and streamline function $\psi(x,y,t)$ satisfy Laplace's equation subject to the following boundary conditions

$$\begin{aligned}
\frac{\partial\phi}{\partial n} &= u(t), \qquad \psi = y\cdot u(t), \qquad \textit{on the wavemaker}\\
\frac{\partial\phi}{\partial n} &= 0, \qquad \psi = 0, \qquad \textit{on the bottom}\\
\frac{D\phi}{Dt} &= \frac{1}{2}|\nabla\phi|^2 - y, \qquad \textit{on the free surface}\\
\frac{Dy}{Dt} &= \frac{\partial\phi}{\partial y}, \quad \frac{Dx}{Dt} = \frac{\partial\phi}{\partial x} + u(t), \qquad \textit{on the free surface}
\end{aligned} \tag{1}$$

This transient nonlinear free-surface problem can be solved by using Longuet-Higgins' idea as follows:

First, the boundary with corner points is conformally mapped onto an upper-half standard plane (see Figs. 1 and 2) with the mapping function given in Eq. (2):

$$z(\zeta) = x + yi = C_0\int_0^{\zeta}\frac{\sqrt{(\zeta-b)}d\zeta}{\sqrt{(\zeta+1)\zeta(\zeta-a)}}\exp\left\{-\frac{1}{\pi}\int_{-\infty}^{-1}\frac{\theta(t)dt}{(\zeta-t)}\right\} \tag{2}$$

where θ is the angle between the tangent at any point of the free-suface and the x-axis. The complex constant C_0, and two real constants, a and b, are determined from the fact that three points $B(x=0,y=h)$, $D(l,0)$ and $E(l,d)$ in the physical plane are mapped onto $B'(\xi=-1,\eta=0)$, $D'(a,0)$ and $E'(b,0)$ in the standard plane, correspondingly. On the free-surface ($\zeta=\xi+\eta i=\xi$), the boundary correspondence function can be written as Eq. (3):

$$\frac{ds_z}{d\xi} = \frac{\sqrt{(b-\xi)}}{\sqrt{(\xi+1)\xi(a-\xi)}} \exp\Big\{ -\frac{1}{\pi} \int_{-\infty}^{-1} \frac{\theta(t)dt}{(\xi-t)} \Big\} \tag{3}$$

Then, the Hilbert transform in the classical theory of analytic functions is applied to solve the mixed boundary value problem in the standard plane (see Fig. 2). It is evident that the streamline function $\psi(x, y, t)$ remains constant (zero) at infinity on the free-surface. According to Ref.[1], there are two solutions (bounded and unbounded) at the separate point ($\xi = -1$) in this mixed boundary value problem. Based on the physical condition, the bounded solution is chosen in our computation. The potential $\phi(\xi)$ on the wavemaker and bottom ($\xi > -1$), and the streamline function $\psi(\xi)$ are obtained from

$$\phi(\xi) = \frac{\sqrt{\xi+1}}{\pi}\Big[\int_{-1}^{0} \frac{\psi(t)dt}{\sqrt{t+1}(t-\xi)} - \int_{-\infty}^{-1} \frac{\phi(t)dt}{\sqrt{-1-t}(t-\xi)}\Big], \quad \xi > -1 \tag{4}$$

$$\psi(\xi) = \frac{\sqrt{-1-\xi}}{\pi}\Big[\int_{-1}^{0} \frac{\psi(t)dt}{\sqrt{t+1}(t-\xi)} - \int_{-\infty}^{-1} \frac{\phi(t)dt}{\sqrt{-1-t}(t-\xi)}\Big], \quad \xi < -1 \tag{5}$$

Finally, a time-stepping process (first order) is applied, in which the first-order finite-difference of free-surface conditions with respect to time t (see Eq. (1)) is used.

Our computations are carried out in the computational plane (see Fig. 3), which is obtained from the conformal mapping of the standard plane. Eqs. (3) , (4) and (5), therefore, are transformed into

$$\frac{ds_z}{dx_c} = \frac{\sqrt{\cosh x_c + \cosh\beta}}{\sqrt{\cosh x_c + \cosh\alpha}} \exp\Big\{\frac{1}{\pi}\int_{0}^{+\infty} \frac{\theta(t)\sinh t dt}{(\cosh x_c - \cosh t)}\Big\} \tag{6}$$

$$\begin{aligned} \phi(y_c) = & \frac{\sqrt{\cos y_c + 1}}{\pi}\Big[\int_0^{\pi} \frac{\psi(t)\sin t dt}{\sqrt{\cos t + 1}(\cos t - \cos y_c)} \\ & + \int_0^{+\infty} \frac{\phi(t)\sinh t dt}{\sqrt{\cosh t - 1}(\cosh t + \cos y_c)}\Big], \quad x_c = 0, \ \ 0 < y_c < \pi; \end{aligned} \tag{7a}$$

$$\begin{aligned} \phi(x_c) = & \frac{\sqrt{\cosh x_c + 1}}{\pi}\Big[\int_0^{\pi} \frac{\psi(t)\sin t dt}{\sqrt{\cos t + 1}(\cos t - \cosh x_c)} \\ & + \int_0^{+\infty} \frac{\phi(t)\sinh t dt}{\sqrt{\cosh t - 1}(\cosh t + \cosh x_c)}\Big], \quad y_c = 0, \ \ x_c > 0; \end{aligned} \tag{7b}$$

$$\begin{aligned} \psi(x_c) = & \frac{\sqrt{\cosh x_c - 1}}{\pi}\Big[\int_0^{\pi} \frac{\psi(t)\sin t dt}{\sqrt{\cos t + 1}(\cos t + \cosh x_c)} \\ & + \int_0^{+\infty} \frac{\phi(t)\sinh t dt}{\sqrt{\cosh t - 1}(\cosh t - \cosh x_c)}\Big], \quad y_c = \pi, \ \ x_c > 0; \end{aligned} \tag{8}$$

In the numerical scheme, the tangential angle of the free-surface, θ, the potential and streamline functions ϕ and ψ are approximated by quadratic elements. The Lagrangian points on the free-surface can be distributed arbitrarily and the free-surface is described by cubic splines in terms of the arc length s of the free surface. The nonlinear algebraic equations of θ which are discretized from Eqs. (2) and (6) are solved by Newton's method. With a good guess of initial values, the mapping function can be found without any difficulties. It is evident that integrations in Eqs. (7) and (8) can be computed very accurately, even though there are integrable singularities.

Three numerical examples are given in Figs. 4-8. Fig. 4a shows the potential error distribution on the wave-maker for the impulsive-motion case at $t = 0^+$ by using BEM based on Lin's Ph.D. thesis. Fig.4b shows the velocity potential error distribution by using our numerical scheme with the same computational parameters. Compared with the analytic and the BEM results, this method gives very accurate values. Fig.5

shows the predicted free-surfaces at 6 different time steps for the impulsive-motion case. As a numerical test, the Lagrangian points are distributed on the free-surface nearly uniformly. It can be seen that the predicted free-surface is in good agreement with the solution of small-time expansion. Fig. 6 shows the wave generated from the wavemaker sinusoidal motion past a flat-bottom by using constant-spacing elements. Results obtained are close to Lin's (1984). Figs. 7-8 show the same waves which are propagating past both a flat-bottom and a step-bottom. The step is located at $x = 5$. It can be seen that the dispersive relation in the linear theory is not exact but it is a good approximation. However, the amplitude relation is not well defined.

Obviously, this numerical scheme is a higher-order method compared with the traditional BEM. The difficulty with spatial singularity can be overcome by using this method. A higher-order numerical scheme than the traditional MEL can be developed in combination of a higher-order time-stepping process. However, in the application with this method, it is difficult to impose the radiation condition. Another problem arises from the fitting of the free-surface. It is not very easy to fit the free-surface in some extreme cases, such as wave breaking, in which another corner point should be indentified.

References

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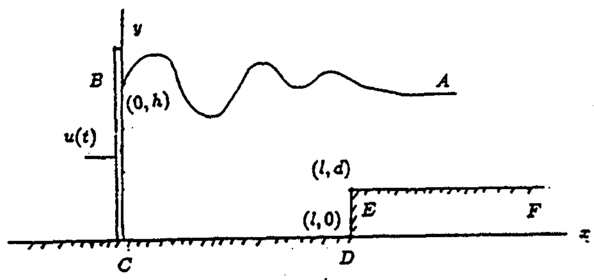

Fig.1 Physical Plane

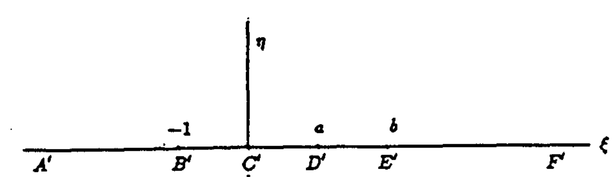

Fig.2 Standard Plane

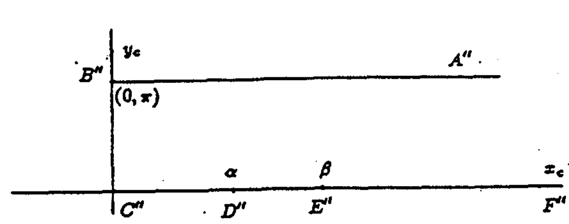

Fig.3 Computational Plane

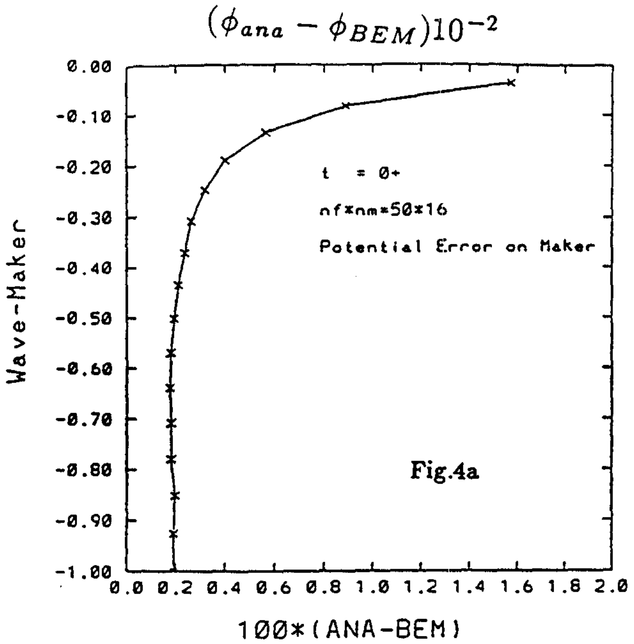

Fig.4a

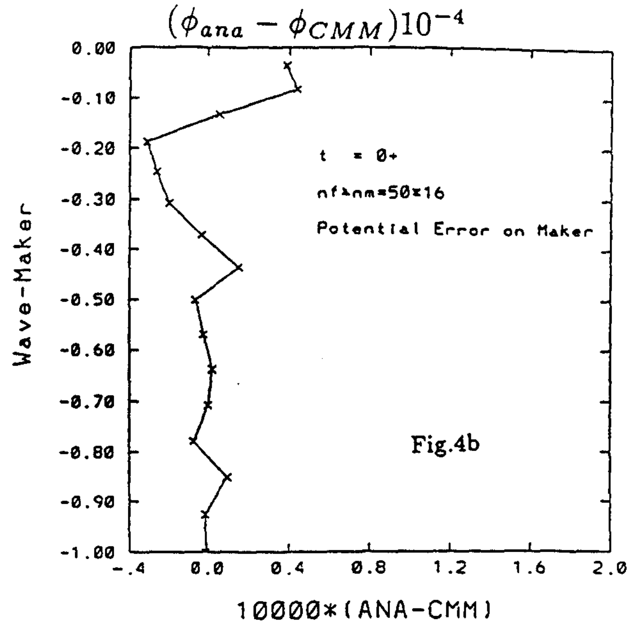

Fig.4b

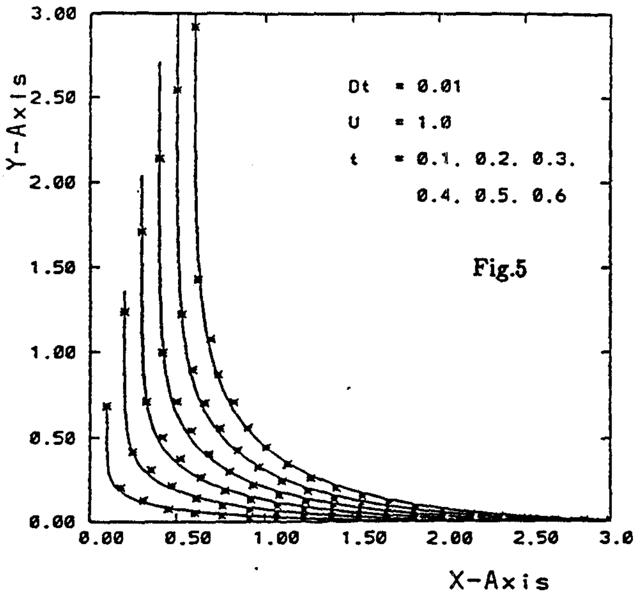

Fig.5

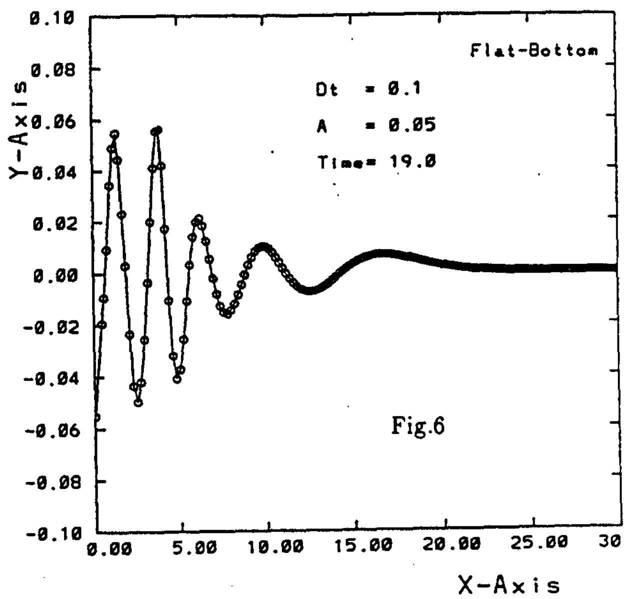

Fig.6

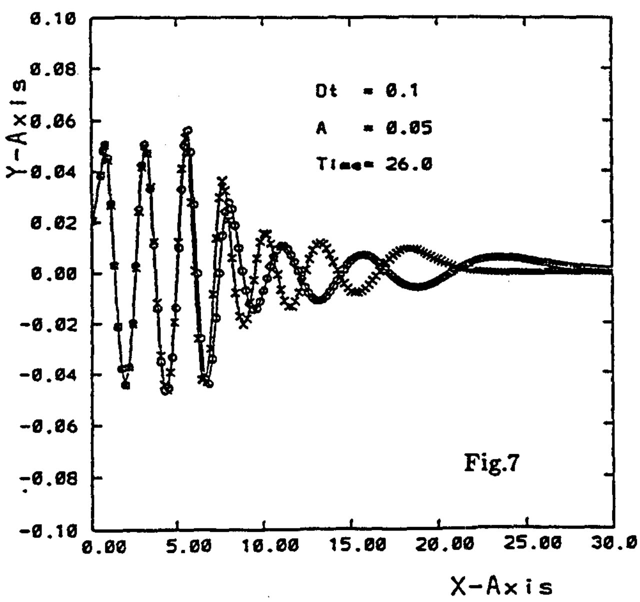

Fig.7

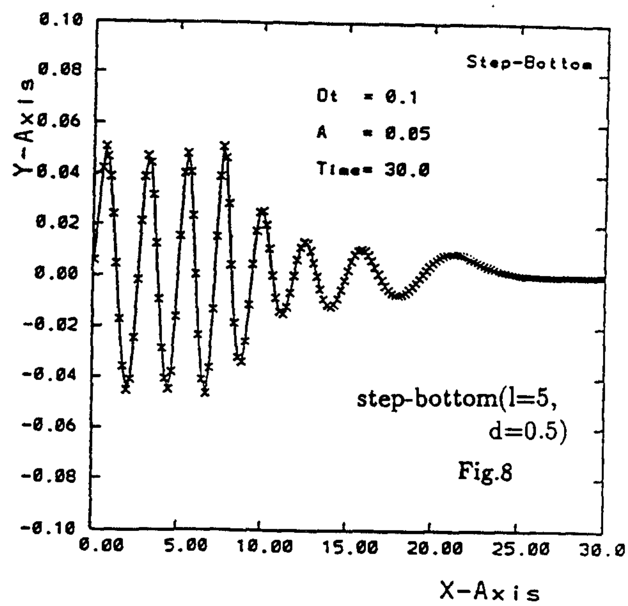

Fig.8

DISCUSSION

YUE: One reason you give is the need for a more satisfactory treatment of the free-surface and body intersection than the mixed Eulerian Lagrangien (MEL) method. It appears that if the flow at this intersection is (weakly) singular (Peregrine), than your use of quadratic and cubic representations for the contact angle and near intersection elevation respective would, as in MEL, be likewise inadequate. Should the flow at this intersection be regular (due to different motions), then both your method and MEL should be equally adequate (assuming comparable representations).
LI & al.: We agree with you. In this paper, we only focused on the spatial singularity. The other singularity which appeared in this problem, time singularity created from the time stepping process, is not dealt with. We expect to develop a real higher-order method in which both spatial and time singularities can be treated numerically.

YEUNG: It appears the improvement in accuracy that you have shown over W.M. Lin's results were due to a higher order boundary element method and were not due to the conformal mapping technique. Also, your present treatment of our free surface definition eliminate one of the nicest feature of the MEL Method, viz, the ability to model highly nonlinear waves. You may like to consider using the arc-length instead of x on the free surface.
LI & al.: We would like to express that we do not quite agree with your opinion. As known, the numerical error in solving nonlinear wave-body interaction with MEL method arises from two sources:

- Time discretization because of the time-stepping process; and,
- Spatial discretization due to BEM.

From our experiences and from Ref.[4], it is evident that the BEM cannot provide accurate results even if there is only one spatial singularity on the boundary. The effect of the usage of the higher-order element in BEM to the treatment of spatial singularity seems negligible. We found that the numerical conformal mapping can successfully deal with spatial singularity no matter what kind of element employed in the numerical scheme.

GREENHOW: I think the principal weakness of this approach is that you need to extrapolate from the free surface near the body to specify the contact angle needed for the mapping. Do you agree?
Breaking waves are in no sense sharp. If you use arc length along the free surface, you should be able to calculate overturning free surfaces and I think you should try it.
LI & al.: The contact angle is very important to the curve-fitting of the free surface near the wave-maker. In our approach, this angle is determined linearly from the contact point and the nearest free-surface point.
Our method is certainly applicable to simulate the breaking waves. In order to do so, we have to improve our numerical scheme specifically in the numerical conformal mapping to handle more complicated geometry. (We are working on it now).

SCHULTZ: In examining the singularity at the contact line of the impulsive wavemaker, it is important to recognize that the small time expansion (and hence standard time stepping schemes) are not uniformly valid at the contact line (see Roberts QJAMM 1988 or Joo, Schultz, Messiter JFM 1990). Hence your comparisons to the "exact" solution with the ln singularity is in some sense a comparison to an artificial problem.

LI & al.: We agree with you. We know that the small time expansion is not valid near the contact line. But a little bit further from the wavemaker, it is supposed to be a very good approximation. Therefore, a higher-order time-stepping process should be applied. However, the comparison in our work is due to the fact that the standard time-stepping process is adopted in our numerical scheme.