Trapped modes above a submerged horizontal plate

by

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It has been proved that at certain frequencies trapped surface-wave modes exist above a submerged long horizontal cylinder of fairly general cross-section, but the proof does not extend to the case of a submerged horizontal plate (Jones [1], p.682). In this paper a numerical technique is developed which provides strong supporting evidence for the existence of trapped modes in this case also. In particular the frequencies of oscillation of such trapped surface-wave modes are approximated by first constructing an inhomogeneous integral equation of the first kind and then converting this into an infinite system of linear algebraic equations, for which the coefficient matrix is positive definite, which can be solved by truncation. It is shown how a long plate approximation can be used to relate these trapped mode frequencies to the reflection coefficient for the problem of a wave travelling above a semi-infinite plate being totally reflected at its end, a problem which can be solved exactly using the Wiener-Hopf technique. This approximation turns out to be extremely accurate for most parameter values and provides a very quick method for the calculation of the trapped mode frequencies.

Formulation

Cartesian axes are chosen with the mean free surface the (x,z)-plane and y measured vertically downwards, the fluid bottom being $y=h$. A fixed thin rigid plate is placed along $y=d<h$, $-a \le x \le a$, $-\infty < z < \infty$. With the usual assumptions of an inviscid, incompressible fluid there exists a velocity potential $\Phi(x,y,z,t)$. It is further assumed that all motion is time-harmonic with angular frequency ω and that the motion is periodic in the z-direction and so assuming the linear theory of irrotational surface waves we can write

$$\Phi(x,y,z,t) = \Re\{\phi(x,y)e^{i(\ell z-\omega t)}\} \tag{1}$$

where ℓ is the wavenumber in the z-direction and $\phi(x,y)$ satisfies

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}-\ell^2\phi \equiv (\nabla^2-\ell^2)\phi = 0 \qquad \text{in the fluid,} \tag{2}$$

with the boundary conditions

$$K\phi+\frac{\partial\phi}{\partial y}=0 \qquad \text{on } y=0, \text{ where } K\equiv\omega^2/g, \tag{3}$$

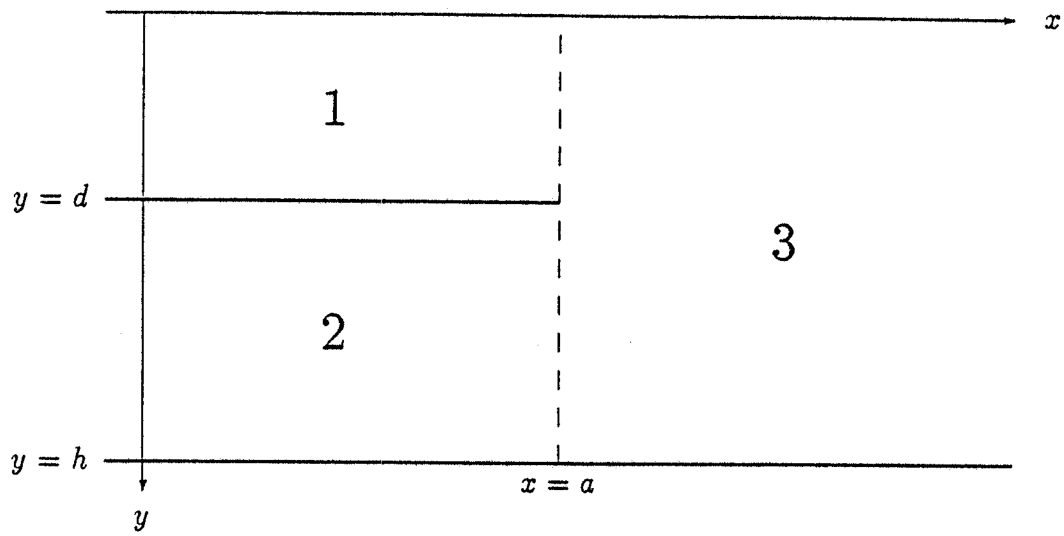

Figure 1: Definition sketch

$$\frac{\partial \phi}{\partial y} = 0 \qquad \text{on } y = h, \tag{4}$$

$$\frac{\partial \phi}{\partial y} = 0 \qquad \text{on } y = d, \ -a \leq x \leq a. \tag{5}$$

For trapped mode solutions we also require

$$\phi, \nabla\phi \to 0 \text{ as } |x| \to \infty. \tag{6}$$

Given a particular geometry therefore, we aim to calculate all the possible pairs of parameters K and ℓ for which a non-trivial solution to equations (2)-(6) exists. This is equivalent to finding a dispersion relation between the wave frequency ω and the longshore wavenumber ℓ. The geometry is symmetric about $x = 0$ and so we can consider solutions that are either symmetric about $x = 0$ or antisymmetric about $x = 0$ and then we need only consider the region $x \geq 0$. This region is then subdivided into three regions as shown in figure 1.

By defining suitable eigenfunctions in the three regions the problem can be formulated as an inhomogeneous integral equation of the first kind with positive definite kernel. This is then converted into an inhomogeneous infinite system of linear algebraic equations with positive definite coefficient matrix which can be solved by truncation. The presence of a singularity in the velocity field at the tip of the plate means that we cannot expect a solution $\{u_n\}$ to this system satisfying $\sum_{n=1}^{\infty} |u_n| < \infty$ and so the proof of convergence of the procedure is not obvious. Results obtained from this method must therefore be treated with some caution.

In a recent paper Evans & Linton [2] showed that trapped mode frequencies for certain problems with bodies in channels could be approximated very accurately by assuming that the region in which waves occur, in this case region 1 above the plate, is a semi-infinite domain. This is a wide-spacing approximation where the ends of the plate are considered to be far enough apart so that sufficiently close to the tip of the plate, $x = a$, it appears that that there is a wave incident from $x = -\infty$ being totally reflected. Such an approximation procedure is not new and it has been applied successfully by many authors including Newman [3] who considered the problem of wave propagation past long two-dimensional obstacles by considering the affects of each end separately. In the case considered here the solution requires the Wiener-Hopf technique to solve the problem of a wave travelling above a semi-infinite plate being totally reflected at its end to which the trapped mode frequencies are related.

Results calculated using both methods will be presented.

References

[1] JONES, D.S. (1953) The eigenvalues of $\nabla^2 u + \lambda u = 0$ when the boundary conditions are given on semi-infinite domains. *Proc. Camb. Phil. Soc.* **49** , 668-684.

[2] EVANS, D.V. & LINTON, C.M. (1990) Trapped modes in open channels. *J. Fluid Mech.* Accepted for publication.

[3] NEWMAN, J.N. (1965) Propagation of water waves past a long two-dimensional obstacle *J. Fluid Mech.* **23** , 23-29.

Martin: Have you tried to extend the arguments in Jones' paper (1953) to your problem?

Linton: There are number of possible methods that might be used to develop an existence proof for trapped modes above a plate including the functional analysis ideas of Jones or, alternatively, the variational ideas of Ursell. As yet I have not looked in any detail at the question of existence in this case.

Ursell: When the plate is very short it may be possible to show the existence of a symmetrical mode, with a frequency just below the cutoff. If this is so, then the existence for a plate of any length would follow from a variational principle (Ursell, JFM 183, 1987, 421-437). It would also be very interesting if the existence of an anti-symmetrical mode could be shown rigorously by a modification of the Wienef-Hopf method. At present there is no proof of the existence of any anty-symmetrical mode.