The numerical limiting form of 2-D nonlinear gravity waves past a submerged vortex by Finite Process Method (FPM)

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Introduction

In [3], based on continuous mapping, a kind of general numerical scheme for solving nonlinear problems, called Finite Process Method (FPM), has been described and used successfully to compute 2-D steep progressive gravity waves in shallow water. The results agree well with those given by Cokelet (1977), Schwartz (1974) and Rienecker & Fenton (1981).

As mentioned in [3], the FPM can avoid iterative techniques so that it is insensitive to the initial solutions. Generally, accurate results can be obtained if the domain of the mapping $p \in [0,1]$ is finely discretized, i.e., Δp is small enough. But more CPU time is needed for a smaller Δp.

Salvesen and von Kerczek [2] gave numerical solutions of 2-D gravity waves in finite water-depth generated by uniform flow over a submerged vortex with circulation τ in cases $|\tau/2\pi| \leq 3.2$ (ft^2/sec) and vortex-submergence $b = 4.5$ ft. They compared their numerical results with perturbation solutions at third-order approximation for *deep* waves; but it seems difficult for their numerical scheme to treat the infinite water-depth.

In the present work, the FPM together with the Boundary Element Method (BEM) are applied to compute the 2-D gravity waves past a submerged vortex with circulation τ. The full nonlinear free surface conditions are satisfied at the wave contour. Water depth is infinite. Simple sources ($\ln r$) are distributed on a line above the water surface. The method given in [1] is used to satisfy the radiation condition.

Basic ideas of Finite Process Method :

Generally, steady nonlinear gravity waves can be described as follows:

$$\nabla^2 \phi(x,y,z) = 0 \qquad \text{in } \Omega \tag{1}$$

with boundary conditions:

$$R(\phi) = g\phi_z + \frac{1}{2}\nabla\phi\nabla(\nabla\phi\nabla\phi) = 0 \qquad \text{on } z = \zeta(x,y) \tag{2}$$

$$\zeta(x,y) = \frac{1}{2g}(U^2 - \nabla\phi\nabla\phi) = H(\phi) \qquad \text{on } z = \zeta(x,y) \tag{3}$$

$$\left.\frac{\partial\phi}{\partial n}\right|_{\partial B} = 0 \tag{4}$$

where, $\phi(x,y,z)$ is the velocity potential function, $\zeta(x,y)$ is the wave-elevation, U is the velocity of body ∂B, g is the gravity acceleration. The coordinate-system $o - xyz$ with z positive upwards is moving with the same velocity U of the body ∂B.

In order to solve above nonlinear problem, we construct a continuous mapping $\phi(x,y,z) \rightarrow \overline{\phi}(x,y,z;p)$, $\zeta(x,y) \rightarrow \overline{\zeta}(x,y;p)$ and $\Omega \rightarrow \overline{\Omega}(p)$ as follows:

$$\nabla^2 \overline{\phi}(x,y,z;p) = 0 \qquad \text{in } \overline{\Omega}(p) \tag{5}$$

with boundary conditions:

$$R(\overline{\phi}) = (1-p)R(\phi_0) \qquad \text{on } z = \overline{\zeta}(x,y;p) \tag{6}$$

$$\overline{\zeta}(x,y;p) = pH(\overline{\phi}) + (1-p)\zeta_0(x,y) \qquad \text{on } z = \overline{\zeta}(x,y;p) \tag{7}$$

$$\left.\frac{\partial\overline{\phi}(x,y,z;p)}{\partial n}\right|_{\partial B} = 0 \tag{8}$$

where $p \in [0,1]$ and $\phi_0(x,y,z)$ [§], $\zeta_0(x,y)$ are the initial solutions which are obviously the solutions of the Eq.(5)~(8) at $p=0$. For simplicity, call Eq.(5)~(8) *zero-order process equations*, which are the same at $p=1$ as the original equations (1)~(4) so that we have the relation:

$$\phi(x,y,z) = \phi_0(x,y,z) + \int_0^1 \overline{\phi}^{[1]}(x,y,z;p)\,dp \tag{9}$$

$$\zeta(x,y) = \zeta_0(x,y) + \int_0^1 \overline{\zeta}(x,y;p)\,dp \tag{10}$$

where

$$\overline{\phi}^{[1]}(x,y,z;p) = \frac{\partial\overline{\phi}(x,y,z;p)}{\partial p} \qquad \overline{\zeta}(x,y;p) = \frac{\partial\overline{\zeta}(x,y;p)}{\partial p}$$

which satisfy following euqations obtained by deriving Eq.(5)~(8) about p :

$$\nabla^2\overline{\phi}^{[1]}(x,y,z;p) = 0 \qquad \text{in } \overline{\Omega}(p) \tag{11}$$

with boundary conditions:

$$g\overline{\phi}_z + \frac{1}{2}\nabla\overline{\phi}^{[1]}\nabla(\nabla\overline{\phi}\nabla\overline{\phi}) + \nabla\overline{\phi}\nabla(\nabla\overline{\phi}\nabla\overline{\phi}^{[1]}) - \frac{p\nabla\overline{\phi}\nabla\overline{\phi}^{[1]}\frac{\partial R(\overline{\phi})}{\partial z}}{g+p\nabla\overline{\phi}\nabla\overline{\phi}_z}$$
$$= -R(\phi_0) - \frac{\{H(\overline{\phi})-\zeta_0(x,y)\}\frac{\partial R(\overline{\phi})}{\partial z}}{1+p\nabla\overline{\phi}\nabla\overline{\phi}_z/g} \qquad \text{on } z=\overline{\zeta}(x,y;p) \tag{12}$$

$$\overline{\zeta}^{[1]}(x,y;p) = \frac{H(\overline{\phi})-\zeta_0(x,y)-p\,\nabla\overline{\phi}\nabla\overline{\phi}^{[1]}/g}{1+p\nabla\overline{\phi}\nabla\overline{\phi}_z/g} \qquad \text{on } z=\overline{\zeta}(x,y;p) \tag{13}$$

$$\left.\frac{\partial\overline{\phi}^{[1]}(x,y,z;p)}{\partial n}\right|_{\partial B} = 0 \tag{14}$$

It is of interest that the above equations are linear in $\overline{\phi}^{[1]}(x,y,z;p)$ and $\overline{\zeta}^{[1]}(x,y;p)$ so that no iterative techniques are needed. Runge-Kutta's method should be applied to obtain $\overline{\phi}(x,y,z;p+\Delta p)$ and $\overline{\zeta}(x,y;p+\Delta p)$. Note that the solutions are obtained at $p=1$.

Some numerical results

As a simple application of the numerical method described above, the steady nonlinear water waves past a submerged vortex are researched. The velocity of steady inflow is 10 fps, the vortex is submerged 4.5 feet under the undisturbed water surface.

In the similar case, Salvesen and von Kerczek [2] gave the limiting form at $\tau/2\pi = 2.70$ (ft^2/sec) with the maximum ratio of wave height-to-length $(H/\lambda)_{max} = 0.12$, $\zeta_{max} = 0.87 * U^2/2g$, wave-length $\lambda = 18.0$ ft and maximum slope $= 24.4$ degree. We obtain the limiting wave at $\tau/2\pi = 2.50$ (ft^2/sec) with $(H/\lambda)_{max} = 0.125$, $\zeta_{max} = 1.316ft = 0.849 * U^2/2g$, wave-length $\lambda = 17.37$ ft and maximum slope $= 25.8$ degree, which is steeper than that given in [2].

In case of $\tau/2\pi < -3.20$ (ft^2/sec), we obtain also converged results. It is of interest that the first crest in case of $\tau/2\pi = -6.0$ (ft^2/sec) is much higher than those far downstream which are so small that it is

§ $\phi_0(x,y,z)$ satisfies Laplace's equation and the boundary condition $\partial\phi_0/\partial n = 0$ on the body surface ∂B

nearly a solitary wave and the corresponding wave-resistance is nearly zero, shown as Fig.3 and Fig.4. In case $\tau/2\pi = -6.0$ (ft^2/sec), the height of the first crest $\zeta_{max} = 1.372$ (ft) $= 0.885 * U^2/2g$ with the maximum slope $= 21.7$ degree. For $\tau < 0$, the limiting form of wave-elevation is obtained at $\tau/2\pi = -6.49$ (ft^2/sec), with the first crest $\zeta_{max} = 1.459$ ft $= 0.941 * U^2/2g$ and maximum slope $= 29.7$ degree. These results are not reported in [2].

Fig.1 and Fig.2 show the limiting form of elevations in case of positive or negative circulation, respectively. Fig.3 shows the wave-elevation in case of $\tau/2\pi = -6.0$ (ft^2/sec). Fig.4 shows the comparison of wave-resistance to the results given in [2].

It seems that the water depth has a great influence, specially to the wave-length and to the limiting form of wave-elevation.

According to Salvesen's experiment, the breaking of 2-D waves past a submerged body will occur at the first crest. Our numerical result obtained at $\tau/2\pi = -6.49$ (ft^2/sec) supports his conclusion.

Discussion and Conclusions

It seems easy for Finite Process Method to satisfy the two nonlinear free surface conditions on the wave-elevation. It seems that the gravity waves close to the limit state can be numerically researched by Finite Process Method, because it is not needed for FPM to use iterative techniques and so the numerical scheme is more insensitive to the initial solutions than iterative models. This is an advantage of Finite Process Method and one can apply this numerical scheme to slove 3-D gravity wave problems, specially those with great Froude numbers. And it seems also that the method given in [1] for treatment of radiation condition is simple and efficient for our numerical scheme.

In case $\tau/2\pi = -6.0$ (ft^2/sec), it is nearly a solitary wave and its corresponding wave-resistance is nearly zero. Has this result any practical meanings in engineering?

It seems doubtful whether or not the perturbation solution at higher-order approximation can give the corresponding results, because perturbation method is based on small parameters and is, strictly speaking, just only a kind of approximate analytical method.

Acknowledgements

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Reference

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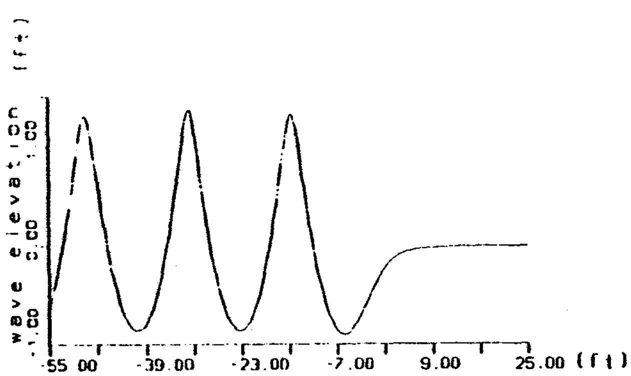

Figure 1. Steepest wave elevation in case of positive circulation
(inflow velocity U=10 fps, vortex submergence b = 4.5 ft)

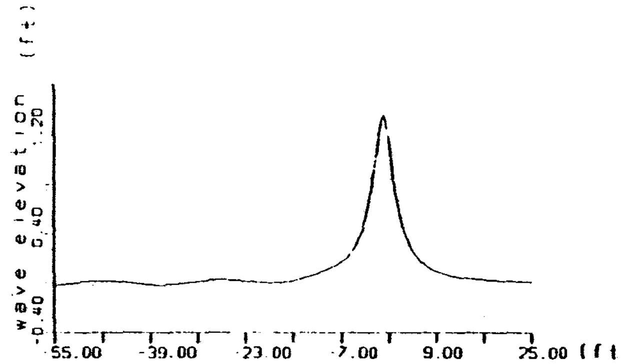

Figure 3: Wave elevation in case of $\Gamma/2\pi$ = -6.0
(inflow velocity U=10 fps, vortex submergence b = 4.5 ft)

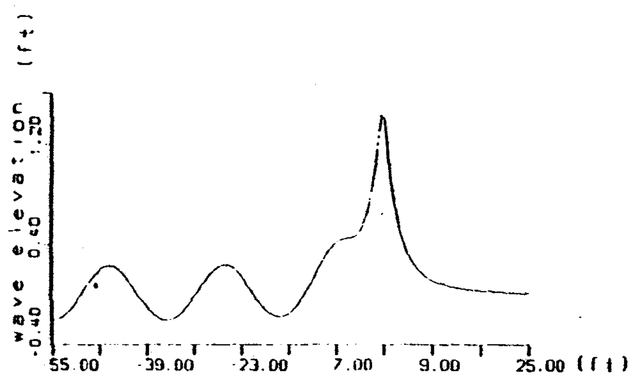

Figure 2: Steepest wave elevation in case of negative circulation
(inflow velocity U=10 fps, vortex submergence b = 4.5 ft)

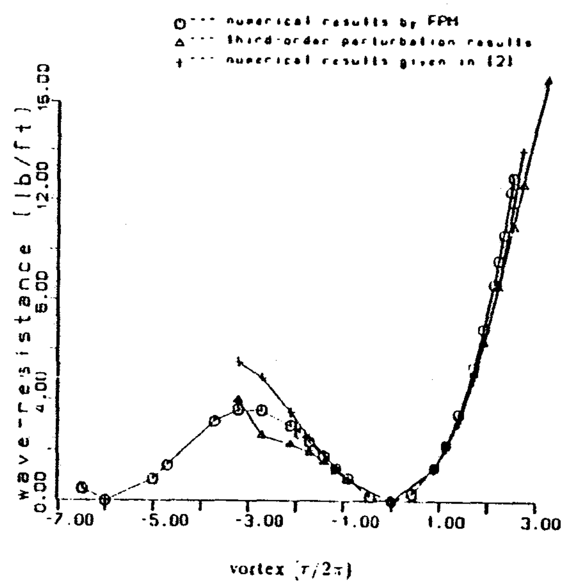

Figure 4: Wave resistance of a submerged vortex
(inflow velocity U=10 fps, vortex submergence b = 4.5 ft)

Yeung: In the last set of results that you showed, you indicated the possibility of zero wave resistance for a finite-strength vortex in a uniform stream. L. Forbes studied this problem for a variety of vortex strengths and Froude numbers. The work was published in J.Eng. Math in the 80's. I don't believe that he found a single incidence of vanishing drag. What physical mechanism do you suppose could lead to this behavior for a single vortex?

Liao: Thank you very much for your reference. Unfortunately, I have not read the paper. I obtained my results for the case where the two nonlinear free-surface boundary conditions are satisfied quite exactly (nondimensional error $< 10^{-10}$). I also used another iterative model, which can give the same results for the case when $\tau/2\pi = -6.0$. I will think about your question deeply.

Liu: What is the maximum surface wave slope to which your method can be applied? (Compared to the critical Stoke's wave slope, for example.) Why?

Liao: The suppositions that we need are that $\phi(x, y, z; p)$, $\zeta(x, y; p)$, $\phi^{(1)}(x, y, z; p)$ and $\zeta^{(1)}(x, y; p)$ exist for $p \in [0, 1]$. If these suppositions are not satisfied for cases of very strong nonlinearity, I would think that the method cannot be used.