Computation of Nonlinear 2-D Free-Surface Flow Using the Hilbert Method

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The problem studied here is concerned with the 2-D, steady, irrotational and free-surface flow. The fluid is assumed to be incompressible and inviscid. This nonlinear problem could be solved by various approaches. An alternative was given by Smith and Abd-el-Malek (1983) using the so-called Hilbert method to solve a problem of waterfall flow. Boutros, et al.(1987) employed this method for the nonlinear solution of a 2-D flow past a triangular obstacle in a stream bed, but only the solutions with large depth Froude numbers were presented and their numerical results seem unreasonable. We improved Boutros' work by using a new numerical scheme and extended this method in application to other bottom configurations.

The bottom topography is regarded as a general polygon as shown in Fig.1. Following Boutrous' derivation, the natural logarithmic transformation is applied to map the infinite strip region in the complex potential plane (see Fig.2) onto the upper half-plane of the auxiliary $t-$ plane (see Fig.3). Then, the Hilbert transforms (see Gakhov,1966) and free-surface Bernoulli equation are employed to establish a system of nonlinear integral equations for the free-surface angle $\theta_f(t)$, the bottom angle $\theta_b(t)$, the fluid speeds $q_f(t)$ on the free-surface and $q_b(t)$ on the bottom as follows,

$$\theta_f(t) = \frac{\sqrt{t-1}}{\pi}\Big\{\int_{-\infty}^{1}\frac{-\theta_b(s)}{(s-t)\sqrt{1-s}}ds + \int_{1}^{\infty}\frac{\ln q_f(s)}{(s-t)\sqrt{s-1}}ds\Big\}, \quad t>1. \tag{1}$$

$$q_f^3(t) = 1 - \frac{3}{\pi F^2}\int_t^{\infty}\frac{\sin\theta_f(s)}{(s-1)}ds, \quad t>1. \tag{2}$$

$$\ln q_b(t) = \frac{\sqrt{1-t}}{\pi}\Big\{\int_{-\infty}^{1}\frac{-\theta_b(s)}{(s-t)\sqrt{1-s}}ds + \int_{1}^{\infty}\frac{\ln q_f(s)}{(s-t)\sqrt{s-1}}ds\Big\}, \quad t<1. \tag{3}$$

where all variables are dimensionless and $\theta_b(t)$ is known.

The numerical solution can be found from Eqs.(1) and (2) by a simple iterative scheme if the images in the $t-$plane(see Fig.3) of the vertices at the bottom are given. The first integrals in Eqs.(1) and (3) can be obtained analytically and the singularity in the second integral of Eq.(1) can be removed. We found that the trigonometric transformations used by Boutros are not suitable for the practical computation. The exponential transformation, which is the reverse of the original logarithmic transformation, was applied. This means the computation are carried out in the infinite strip of the complex potential $W-$plane. Eqs.(1) , (2) and (3) become

$$\theta_f(\phi) = -\frac{2}{\pi}\sum_{i=1}^{i=N-1}\alpha_i\tan^{-1}\Big(\frac{G_i e^{-\frac{\pi}{2}\phi}}{1+e^{-\pi\phi}+G_i'}\Big) + \int_{-\infty}^{+\infty}\frac{\ln q_f(\kappa)-\ln q_f(\phi)}{e^{\frac{\pi}{2}(\phi-\kappa)}-e^{\frac{\pi}{2}(\kappa-\phi)}}d\kappa. \tag{4}$$

$$q_f^3(\phi) = 1 - \frac{3}{F^2}\int_{-\infty}^{\phi}\sin\theta_f(\kappa)d\kappa. \tag{5}$$

$$\ln q_b(\kappa) = -\frac{1}{\pi}\sum_{i=1}^{i=N-1}\alpha_i\ln\Big(\frac{1-e^{-\pi\phi}+G_i'+G_ie^{-\frac{\pi}{2}\phi}}{1-e^{-\pi\phi}+G_i'-G_ie^{-\frac{\pi}{2}\phi}}\Big) + \int_{-\infty}^{+\infty}\frac{\ln q_f(\kappa)d\kappa}{e^{\frac{\pi}{2}(\phi-\kappa)}+e^{\frac{\pi}{2}(\kappa-\phi)}}. \tag{6}$$

where N is the number of the polygnal corners, $G_i = e^{-\frac{\pi}{2}\phi_{i+1}} - e^{-\frac{\pi}{2}\phi_i}$, $G_i' = e^{-\frac{\pi}{2}(\phi_i+\phi_{i+1})} - 1$, $\phi_1 = 0$, $\phi_2 \ldots, \phi_N$ are the corresponding points in the W−plane of the bottom polygonal corners in the physical plane, and the definition of α_i is refered to Fig.1.

The free surface and the bottom configuration can be obtained from

$$\begin{cases} x_f(\phi) = x_0 + \int_{-\infty}^{\phi} \dfrac{\cos\theta_f(\kappa)}{q_f(\kappa)} d\kappa; \\ y_f(\phi) = 1 + \int_{-\infty}^{\phi} \dfrac{\sin\theta_f(\kappa)}{q_f(\kappa)} d\kappa. \end{cases} \tag{7}$$

$$l_i = \sin\alpha_i \int_{\phi_i}^{\phi_{i+1}} \frac{d\kappa}{q_b(\kappa)}. \tag{8}$$

where x_0 is the coordinate of the truncated point upstream and the definition of l_i is refered to Fig.1.

The values of $\phi_2, \phi_3 \ldots, \phi_N$ (i.e. $t_2, t_3, \ldots, t_N$) are unknown. They can be determined by the hybrid Powell's method due to the fact that the variables $\phi_2, \phi_3 \ldots, \phi_N$ are functions of $l_1, l_2, \ldots l_{N-1}$. Since the accuracy of the numerical solution is mainly dependent on the integrals in Eqs.(4),(5) and (6), the Romberg method with the cubic spline fitting for $\theta_f(\phi)$, $q_f(\phi)$ and $q_b(\phi)$ is adopted to carry out the numerical integrations. Double-precision arithmetic is used in the computation. The initial solution is taken from the nongravity and linear solutions for supercritical and subcritical cases, respectively. The relative error is chosen to be less than 10^{-3} for numerical integration. The iterative process continues until the maximum difference of two successions is less than a pre-designated small number 10^{-6}.

The free-surface flows over triangular obstacles are computed in order to compare with the linear solution and Boutros' results. The computations for the flows over other bottom topographies are also carried out. Figs.4, 5 and 10 show the differences of the results between the linear theory and nonlinear approach for the supercritical and subcritical Froude numbers with different bottom configurations. The linear approach is from Lamb (1932) on the assumption that the solution could be considered as a small perturbation to the uniform flow. For $F^2 < 1$, Lamb's linear theory gives a surface profile without waves upstream, but generally, with regular waves downstream. For $F^2 > 1$, a symmetric wave-free profile is obtained. For the critical flow, the linear solution is unavailable. It is obvious that the nonlinear free-surface profile is almost symmetric and that the nonlinear elevation is much greater than that predicted by the linear theory. For the subcritical flow, the computed nonlinear free-surface profile is wave-free upstream but with a wave train downstream. The downstream wave train seems to possess nonlinear wave characteristics. These conclusions are similar to Forbes' results (1981, 1982), even though the obstructions considered in Forbes' papers are semi-circular and semi-elliptical bodies.

It should be noted that our computed results are very different from that of Boutros. In Figs.6 to 8, we re-compute all cases that Boutros considered. It is found that the flow does not go downwards for $\alpha \geq 1$ *rad* and $F = \sqrt{10}$. This is quite contrary to Boutros' results.

The effect of Froude number on the free-surface profile for one bottom configuration is illustrated in Fig.8. It is clear that in increasing the Froude number, the free-surface elevation decreases. This means that the effect of gravity on the free surface could be neglected for a large Froude number ($F \gg 1$). The linear theory gives the same conclusion, but Boutros gave the opposite result. It is interesting that the steady nonlinear solution exists around the critical Froude number and that the computed nonlinear free-surface profiles shown in Fig.8 do not possess waves downstream for $F^2 \geq 0.9$. In another word, the critical Froude number may not be exactly 1.

In principle, this method could be used to solve the flow with arbitrary bottom configurations. In Figs.9 and 10, the computed free-surface profiles for six bottom configurations are presented. All numerical results seem reasonable. But the results for small Froude numbers are not presented because the convergence of the iteration process is not satisfactory. The convergence of the direct iteration process depends greatly on the value of Froude number F. For the supercritical case ($F > 1$), only three iterations are required. But as the Froude number becomes smaller, the convergence gets worse. The simple iteration may converge only if the initial values of q on the free surface are taken from the linear theory for the subcritical case. It is expected that the adoption of Newton method may improve the convergence of the numerical solution for small Froude number.

References

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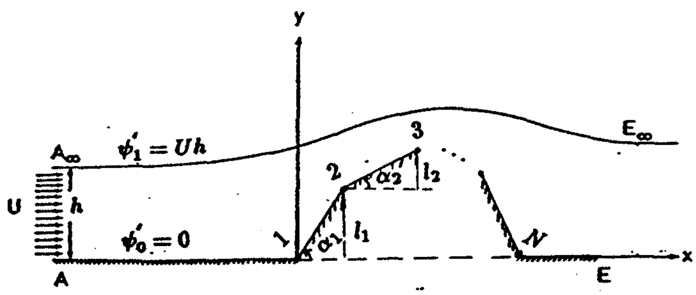

Fig.1 Z-plane , the physical plane

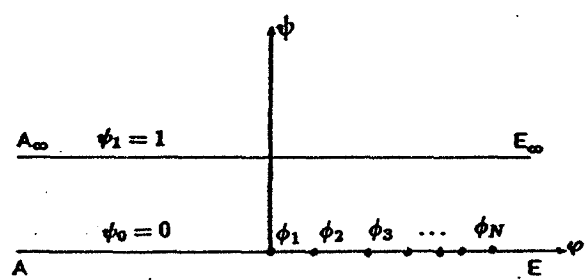

Fig.2 W-plane , the normalized complex potential plane

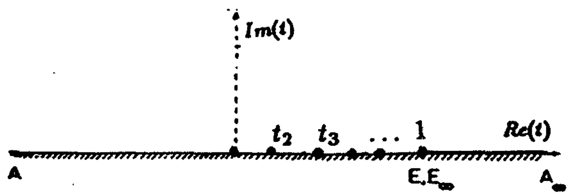

Fig.3 t-plane , the upper half-plane

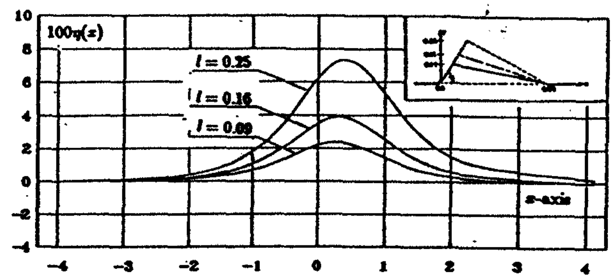

Fig.7 Effect of the bottom height on the free-surface profiles for a fixed $\alpha = 1.0$ ($F^2 = 10$)

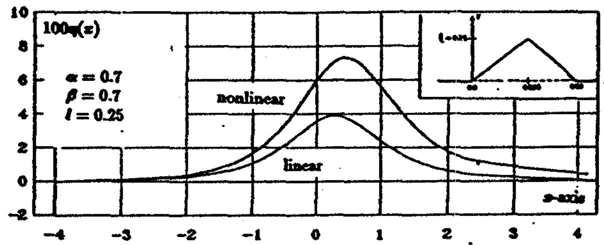

Fig.4 Comparison of the linear and nonlinear free-surface profiles for supercritical case ($F^2 = 10$)

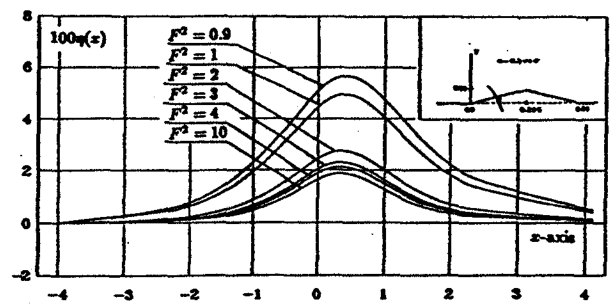

Fig.8 Effect of the Froude number F on the free-surface profiles for $\alpha = 0.3$ $\beta = 0.3$ $l = 0.09$

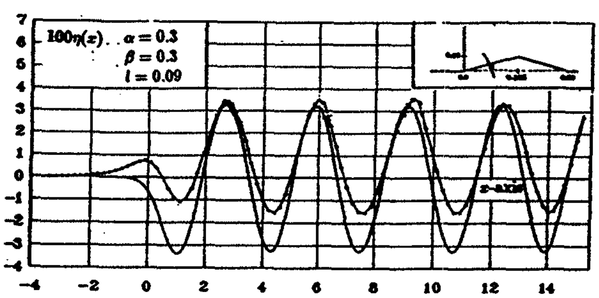

Fig.5 Comparison of the linear and nonlinear free-surface profiles for subcritical case ($F^2 = 0.49$)

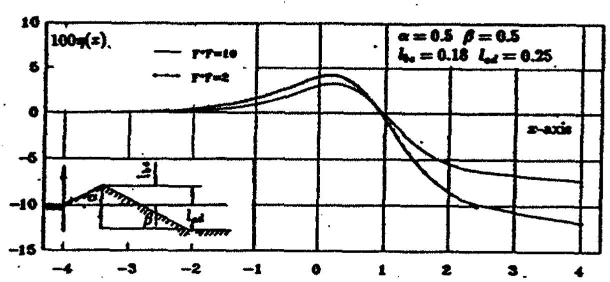

Fig.9 The free-surface profiles for other irregular bottom: Case (1)

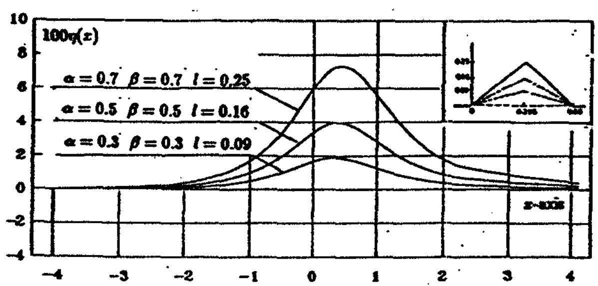

Fig.6 Effect of the bottom height on the free-surface profiles for supercritical case ($F^2 = 10$)

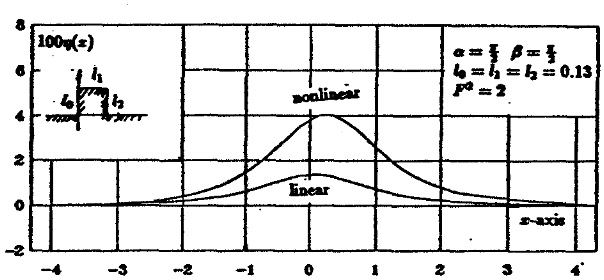

Fig.10 The free-surface profiles for other irregular bottom: Case (2)

Tuck: (1) I would like to add some references. Mainly the name Vanden-Broeck should appear. He has published 10-20 papers in this area. Also, a review and extension article by Dias and Tuck is to appear in JFM this year. (2) The final nonlinear integro-differential equation is essentially the same as the "Nekrasor" equation (9) of my paper at this Workshop. (3) The experience of recent workers such as Dias on such equations is that packaged routines in the "NAG" or "IMSL" libraries are quite satisfactory for solution of the nonlinear algebraic equations that result from discretization. Thence there is no need for the effort to use Newton iteration via computation of the gradient matrix, in cases where the direct iterative method discussed by the authors converges badly.

Li, Chuang, Hsiung: (1) Thank you for your comments and references. This problem is a classic problem which has been studied for a long time. We have only worked in this field for a short period so that some important papers have been overlooked. After the meeting, we reviewed nearly every paper in this area. We found that there are two other methods. One was given by Bloor and King (Refs.[A2][A3]), who also used a conformal mapping method. The other one was presented by Vanden-Broeck, Dias and Tuck (Refs.[A1][A4][A5]) using the "Series Truncation Procedure". Abd-el-Malek's (or Boutros') method was similar to the former but the nonlinear integral equations were different. It should be pointed out that the King and Bloor (Ref.[A3]) also found that their numerical solution was quite different from Boutros' result for one case. However, our numerical results are nearly the same as King's results for triangular obstacles. (2) The equations used in our paper are integrated differential equations because an auxiliary function $H(t)$ was introduced and the formulation was modified so that the Hilbert general solution can be used to establish a system of integral equations, which are not the same as the "Nekrasor" equations. (3) Perhaps you are right. We found that the error comes from using Simpson's rule in the integrations of Eq.(4) and (5). Romberg quadrature might obtain more accurate results. Using Romberg however, makes it difficult to apply other iterative methods (such as Newton's method). So it may be better to formulate Eq.(5) in a differential form.

References

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Tulin: I have a question about the title of your paper. Why is the mathematics used in your analysis called "The Hilbert Method"?

Li, Chuang, Hsiung: Refs. [1] and [2] called this method the Hilbert method since the Hilbert transforms were used to establish a system of nonlinear integral equations.

Cao: You have shown some results for Froude number equal to 1 or near 1, and your method assumes a steady flow. However, it is known that when Froude number is near 1, upstream runaway solitons will be generated periodically and the flow is unsteady.

Li, Chuang, Hsiung: We agree with your comments about the shallow water wave theory, however, in our numerical experiments we do find a steady nonlinear solution around the critical Froude number $F = 1$.