Wave-current interaction with floating bodies at moderate water depth

by John Grue

Department of Mechanics, University of Oslo, Norway

Interaction between ocean waves, a small current and floating bodies in water of finite depth is an important problem within offshore technology. Among the important quantities to predict is the wave drift damping force, which often is the dominant damping mechanism of resonant slow drift motions of floating barges, ships and oil platforms. Earlier 3-D works have treated this theme by assuming the water depth to be infinite, which is a relevant assumption in several practical situations. In the North Sea, on the other hand, where the wave lengths may be up to 300m long and the water depth may be as shallow as 20-50m, which means a ratio between the wave length and the water depth varying between 15 and 6, the effect of the water depth become important for the wave characteristics. This again means that the wave drift forces and the wave drift damping forces may differ considerably from the infinite depth values. As we shall see, a finite water depth is making the effect of a small forward speed even more pronounced than in deep water. The following analysis is a direct generalization of Nossen, Grue and Palm (1991), who considered this problem in deep water.

The boundary value problem

Let us introduce a coordinate system $O-xyz$ with the x- and y-axes in the mean free surface and the z-axis vertical upwards, and let us consider the problem in the frame of reference where the body is oscillating about a mean position and embedded in a uniform current with speed U directed along the negtive x-axis. The fluid is assumed to be homogenous and incompressible, and the motion irrotational. The velocity field may then be governed by a velocity potential Φ^* which satisfies the Laplace equation. Φ^* is composed by a steady potential $U\chi_s$ representing the stationary flow around the body due to the current and an oscillatory part $\Re\phi(\boldsymbol{x})e^{i\sigma t}$ due to incoming and scattered time harmonic waves, as well as oscillatory motions of the body, i.e.

$$\Phi^*(\boldsymbol{x},t) = U\chi_s(\boldsymbol{x}) + \Re\phi(\boldsymbol{x})e^{i\sigma t} \tag{1}$$

Here t denotes time. The steady potential satisfies the rigid wall condition at the free surface (since U is assumed small), at the body and on the sea floor.
Let us decompose the potential ϕ into

$$\phi = \frac{Aig}{\omega}(\phi_0 + \phi_B) \tag{2}$$

where A denotes the incoming wave amplitude, ω the orbital frequency of the incoming waves, ϕ_0 and ϕ_B the incoming and scattered wave potentials, respectively. The incoming wave potential forming a wave angle β with the positive x-axis reads

$$\phi_0 = \frac{\cosh K(z+h)}{\cosh Kh} e^{-iK(x\cos\beta + y\sin\beta)} \tag{3}$$

where K is the wave number of the incoming waves related to the orbital frequency by the dispersion relation

$$\omega^2 = gK \tanh Kh \tag{4}$$

The frequency of encounter σ is related to ω, K and U by

$$\sigma = \omega - UK \cos\beta \tag{5}$$

The boundary condition for $\phi = \phi_0 + \phi_B$ at the free surface reads

$$-\nu\phi + 2i\tau\nabla_1\phi \cdot \nabla_1\chi_s + i\tau\phi\nabla_1^2\chi_s + \frac{\partial\phi}{\partial z} = 0 \text{ at } z = 0 \tag{6}$$

where ∇_1 denotes the horizontal gradient, $\nu = \sigma^2/g$ and $\tau = U\sigma/g$. Far away from the body, $\chi_s \to -x$, and the free surface boundary condition (6) simplifies to

$$-\nu\phi - 2i\tau\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial z} = 0 \text{ at } z = 0 \tag{7}$$

On the sea floor the rigid wall condition applies

$$\frac{\partial\phi}{\partial z} = 0 \text{ at } z = -h \tag{8}$$

Far away from the body, the radiation conditions state that the scattered wave potential behave as outgoing waves.

The Green function

The boundary value problem for ϕ_B is solved by applying Green's second identity to the entire fluid domain. As Green function we apply a pulsating source in a small, uniform current which is satisfying the free surface condition (7) and the proper radiation conditions. This function is given by

$$G(\boldsymbol{x}, \boldsymbol{\xi}) = \frac{1}{r} - \frac{1}{r_2} + \Psi(\boldsymbol{x}, \boldsymbol{\xi}) \tag{9}$$

with $r = |\boldsymbol{x} - \boldsymbol{\xi}|$ and $r_2 = |\boldsymbol{x} - (\xi, \eta, -\zeta - 2h)|$. The wave part of the source potential is given by

$$\Psi(\boldsymbol{x}, \boldsymbol{\xi}) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} B(k,\alpha)\frac{2e^{-kh}}{\cosh kh}\cosh k(\zeta + h)\cosh k(z + h) \times e^{ik((x-\xi)\cos\alpha + (y-\eta)\sin\alpha)}dk\,d\alpha \tag{10}$$

where

$$B(k,\alpha) = \frac{\nu + k(1 - 2\tau\cos\alpha)}{k(\tanh kh + 2\tau\cos\alpha) - \nu} \tag{11}$$

and $\nu = \sigma^2/g$ and $\tau = U\sigma/g$. The path of integration is above the pole $k = k_1$ given by

$$k_1(\tanh k_1 h + 2\tau\cos\alpha) - \nu = 0 \tag{12}$$

For small values of τ we may expand B in a power series by

$$B(k,\alpha) = B^0(k,\alpha) + \tau B^1(k,\alpha) + O(\tau^2) \tag{13}$$

where

$$B^0(k,\alpha) = \frac{\nu + k}{k\tanh kh - \nu} \tag{14}$$

$$B^1(k,\alpha) = 2i^2k\cos\alpha\left(\frac{1}{k\tanh kh - \nu} + \frac{(\nu+k)}{(k\tanh kh-\nu)^2}\right) \quad (15)$$

For small forward speed we thus have

$$G = G^0 + \tau G^1 + O(\tau^2) \quad (16)$$

where G^0 denotes the Green function for $\tau = 0$ (but keeping ν). It is easily seen that

$$G^1 = 2i\frac{\partial^2 G^0}{\partial\nu\partial x} \quad (17)$$

The far-field behaviour of G is obtained by applying contour integration and the method of stationary phase, giving

$$G(R,\theta,z;\xi,\eta,\zeta) = R^{-1/2}\hat{h}(\xi,\theta)\frac{\cosh k_1(\theta)(z+h)}{\cosh k_1(\theta)h}e^{ik_1(\theta)R\cos(\alpha_0-\theta)} \quad (18)$$

Let us introduce the orbital frequency $\omega_1 = (gk_1\tanh k_1h)^{\frac{1}{2}}$ and the group velocity of the scattered waves $c_g^h(\omega_1) = \frac{\partial\omega_1}{\partial k_1}$, and also the group velocity of deep water waves with orbital frequency ω_1, i.e. $c_g^\infty(\omega_1) = \frac{1}{2}\frac{g}{\omega_1}$. The amplitude $\hat{h}(\xi,\theta)$ is then given by

$$\hat{h}(\xi,\theta) = \sqrt{\frac{2\pi}{k_0}\frac{k_1(\tanh k_1h+1)}{c_g^h(\omega_1)/c_g^\infty(\omega_1)}}(e^{k_1\zeta} + e^{-k_1(\zeta+2h)})e^{-ik_1(\xi\cos\alpha_0+\eta\sin\alpha_0)-i\pi/4} + O(\tau^2) \quad (19)$$

and k_0 is the solution of

$$\nu = k_0\tanh k_0h \quad (20)$$

α_0 and $k_1(\theta)$ are, to first order in τ, obtained as

$$\alpha_0 = \pi + \theta - 2\tau^h\sin\theta \quad (21)$$

$$k_1(\theta) = K(1 + 2\tau^h(\cos\theta - \cos\beta)) \quad (22)$$

where

$$\tau^h = \frac{\tau}{\tanh Kh + \frac{Kh}{\cosh^2 Kh}} = \frac{\tau}{c_g^h(\omega)/c_g^\infty(\omega)} \quad (23)$$

The results for α_0 and k_1 are identical to the deep water results except that τ is replaced by τ^h.

Solution of the boundary value problems

By expansion of the velocity potential ϕ in power series of τ, i.e.

$$\phi = \phi^0 + \tau\phi^1 + ... \quad (24)$$

we may obtain ϕ_B at the body surface. For the pure diffraction problem we obtain the following equations which are identical to the infinite depth case except that the Green function is different, i.e.

$$2\pi\phi^0 + \iint_{S_B}\phi^0\frac{\partial G^0}{\partial n}dS = 4\pi\phi_0 \quad (25)$$

$$2\pi\phi^1 + \iint_{S_B}\phi^1\frac{\partial G^0}{\partial n}dS = 2i\iint_{S_F}\phi^0(\nabla_1 G^0\cdot\nabla_1\chi + \frac{1}{2}G^0\nabla_1^2\chi)dS - \iint_{S_B}\phi^0\frac{\partial G^1}{\partial n}dS \quad (26)$$

where S_B denotes the wetted surface of the body and S_F the free surface. The equations for the radiation problems are similar.

The mean horizontal drift force

The x-component of the mean drift force (i.e. in the current direction) is given by

$$F_x = \rho \int_0^{2\pi} \left\{ -\frac{1}{2g}\left[\overline{\left(\frac{\partial \Phi}{\partial t}\right)^2} - U^2 \overline{\left(\frac{\partial \Phi}{\partial x}\right)^2} \right]_{z=0} \cos\theta + \int_{-h}^{0} \left[\frac{1}{2}\overline{|\nabla\Phi|^2} \cos\theta - \overline{\frac{\partial \Phi}{\partial x}\frac{\partial \Phi}{\partial R}} \right] dz \right\} R d\theta \tag{27}$$

which is valid for arbitrary current speed U and water depth h. Inserting $\Phi = \Re e^{i\sigma t} A \frac{ig}{\omega}(\phi_0 + \phi_B)$ with ϕ_0 given by (3) and ϕ_B given by

$$\phi_B = R^{-1/2} H(\theta) \frac{\cosh k_1(\theta)(z+h)}{\cosh k_1(\theta)h} e^{-ik_1(\theta)R(1+O(\tau^2))} + O(\frac{1}{R}) \quad R \to \infty \tag{28}$$

The amplitude function $H(\theta)$ is given as integrals over S_B and S_F of products of $\hat{h}$ and $\phi^0 + \tau\phi^1$. By averaging with respect to time and applying the method of stationary phase, we obtain

$$\frac{F_x}{\rho g A^2} = -\frac{1}{4}\frac{k_0}{\omega^2/g}\left\{ \int_0^{2\pi} \frac{c_g^h(\omega_1)}{c_g^\infty(\omega_1)}(\cos\theta + 2\tau^h \sin^2\theta)|H(\theta)|^2 d\theta + 2\frac{c_g^h(\omega)}{c_g^\infty(\omega)} \cos\beta \Re(S) \right\} + o(\tau) \tag{29}$$

where

$$\theta^0 = \beta + 2\tau^h \sin\beta \tag{30}$$

$$S = \sqrt{\frac{2\pi}{k_0}} e^{i\pi/4} H^*(\theta^0) \tag{31}$$

A star denotes complex conjungate. We note that $\frac{c_g^h}{c_g^\infty}$ approaches unity, $\tau^h \to \tau$, and $\frac{gk_0}{\omega^2} \to \frac{\nu}{K}$ as $h \to \infty$. The ratio between the group velocity in shallow and deep water thus gives the main effect of the shallow water on the combined wave current interaction on floating bodies.

Reference

Nossen, J., Grue, J. and Palm, E. Wave forces on three-dimensional floating bodies with small forward speed. J. Fluid Mech. 227, 1991, pp. 135-160.

Cao: What is the range of Froude numbers (based on the depth)? Is it possible that the Froude number becomes close to the critical value 1? In that case, solitary waves are generated and the nonlinearity of the free surface is very important.

Grue: The Froude number based on water depth varies typically from zero to $O(0.1)$.

Wu: In our work (Wu, & Eatock Taylor, JFM, VOL. 211, pp. 333-353, 1990), we mentioned that ϕ^1 in equation (24) of your paper tends to infinity at large distance. How can you then calculate the second order force using this far field equation?

Grue: The small τ asymptotic expansion of the velocity potential does diverge as $R \to \infty$. In our work, we apply this expansion locally at the body, *i.e.*, in a bounded region, where it gives the leading order contribution to the velocity potential for $\tau \to 0$. When the far field quantities are computed we apply the exact expansion of the Green function and the velocity potential, with no restriction on τ. The far field amplitude of the potential then appears as integrals over the body surface and a limited part of the free surface around the body. This procedure is outlined in detail in Nossen, Grue & Palm (1991) for the infinite water depth case. The same procedure applies when the water depth is moderate.

Zhao: In the case of finite water depth, have you checked whether or not the second order potential contributes to the mean wave drift force damping?

Grue: Provided that there is no circulation in the fluid, there is no contribution from the mean second order potential to the wave drift force damping. (see Grue & Palm, 1990 "Mean forces on floating bodies in waves and currents"; Abstracts: 5th International Workshop on Water Waves and Floating Bodies, Manchester, U.K., ed. P. Martin, 1990)