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Highly oscillatory behaviors in the Neumann-Kelvin problem

by
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With recent interests in the ability of detecting ship wakes, the determination of the wave pattern far behind a ship — also useful for wave-resistance predictions — has become crucial. This new application requires to evaluate with accuracy the high frequency part of the wave spectrum that only among available methods, the Neumann-Kelvin approach can provide, other methods being limited in space or resolution. However, this method, despite its rather long history, has failed, as well as the others, to produce results for ships.

Since 1960 [1], it is known that the Kelvin wave-source potential has an increasingly high oscillatory behavior with increasing amplitude as the field point approaches the track of the source point when both are on the free-surface. It is also known, since 1972 [2], that the formulation of the Neumann-Kelvin problem involves an integral along the waterline — the *line integral* — as well as an integral on the wetted hull surface.

Nevertheless "Neumann-Kelvin" codes avoiding the consideration of the line integral, and therefore of the potential at its most complex regime, have been developed. In view of the wide discrepancies in the results generated by such codes, one could question even more the validity of disgarding the line integral. Of course it can be argued that numerical procedures and approximations could be at fault, or that mathematical difficulties inherent to the problem are responsible, or even that the problem might well be ill-posed. But prior to trying to consider this last possibility, it seems recommendable not to overlook the most obvious ones.

It is only recently that systematic numerical studies of the Neumann-Kelvin problem have been made ([3], [4]). In the same effort to obtain results but departing from the classical Neumann-Kelvin formulation, an approach based on wave-spectrum functions has also been proposed [5]. Simultaneously and under this new interest in the short wavelength part of the wakes, analytical expressions and derived numerical procedures have been developed to evaluate accurately the potential throughout its domain of definition ([6], [7], [8]).

The present work

In the same spirit, it is legitimate to initiate a rigorous study of the problems encountered when using the discretized formulation of the Neumann-Kelvin problem. We start here by trying to clarify the question of the contribution of the highly oscillatory and singular behavior of the Kelvin wave-source potential to the line and the hull integrals.

The contribution of the far-field component — or single integral part — φ_F of the Green function, which accounts for its highly oscillating and singular behavior, to the perturbation potential at a point P in the fluid domain is given by:

$$\phi_F(P) = \int\int_{S_H^{\leq}} \sigma(P')\varphi_F(P,P')\,dS(P') + \frac{1}{k_0}\int_{C_H^{\leq}} n_X(P')\sigma(P')\varphi_F(P,P')\,dY'$$

where the X-axis is in the direction of motion, the Z-axis pointing upward, the Y-axis to port; $S_H^{\leq}$ and $C_H^{\leq}$ are the portions of the wetted hull and the waterline, upstream of P, $n_X(P')$ the X-component of the normal to the hull at P'.

Particular cases of a segment and a flat panel

Consider the cases were $S_H^{\leq}$ and $C_H^{\leq}$ are reduced to a panel $\mathcal{P}$, with one edge in the plane of the free-surface, and a segment $\mathcal{S}$ in the plane of the free-surface, both parametered by $[0,1]^2$ and $[0,1]$ respectively.

Using the following representation for φ_F:

$$\varphi_F(P,P') = -\frac{1}{\pi}\Im\{\int_\Gamma \exp[\chi(\vec{x},u)]\cdot\cosh u\,du\}$$

where $\chi(\vec{x},u) = \frac{z}{2}(1+\cosh 2u) + i\frac{y}{2}\sinh 2u + ix\cosh u$

with $x = k_0(X-X')$, $y = k_0(Y-Y')$, $z = k_0(Z+Z')$ with $k_0 = g/U^2$,

and where Γ is a path joining $-\infty + i\beta$ to $+\infty + i\beta$ in the complex plane (β depending on the respective positions of P and $\mathcal{S}$, or $\mathcal{P}$)

we obtain integral expressions where the complex integration is first to be performed, followed then by the integration over $[0,1]$ or $[0,1]^2$.

We then treat *separately* the contributions from $\mathcal{S}$ and from $\mathcal{P}$. (To stay simple we will consider only the case of $\mathcal{S}$ hereafter.)

Interchange of the order of integrations

Under this form, we interchange the order between the spatial integration and the integration in u, *without any justification* (as everybody does...), and therefore are dealing with an integral of the form:

$$\int_\Gamma\int_0^1 \sigma(s)\exp[\chi(\vec{x}(s),u)]\,ds\ \cosh u\,du$$

Integration by parts in space

By assuming that σ is of class C^k in [0,1] and performing successive integration by parts in s, the contribution from $\mathcal{S}$ reduces to a sum of integrals of the form:

$$\sigma^{(n)}(s)\int_\Gamma \exp[\chi(\vec{x}(s),u)]\frac{\cosh u}{(\frac{d}{ds}[\chi(\vec{x}(s),u)])^{n+1}}\,du \qquad 0 \leq n \leq k,\ \ s = 0,1$$

and a 'non-integrated' term involving $\sigma^{(k)}(s)$. (The path Γ must have been chosen *a priori* to lie above or below the two zeros — which depend on $\mathcal{S}$ and P only — of the denominator in the region $]-\infty,+\infty[\times[-i\pi,+\pi[$, so that they are excluded of any closed contour considered subsequently.)

Asymptotic analysis of the different terms

From there we perform an analysis similar to the ones done for φ_F ([6],[8]), and show that each integral, say L, can be decomposed into:

$$L = \overline{L} + \tilde{L} + \epsilon_L(M, x/\rho)$$

where:

- $\overline{L}$ is a truncated series involving the Bessel functions $I_m(\rho/2)$
- $\tilde{L}$ accounts for the highly oscillatory behavior of L
- $\epsilon_L \to 0$ as M and $|x|/\rho$ get large (ρ and α being defined by $\rho e^{i\alpha} = -z + iy$.)

for (x, y, z) in a region of the kind:

$$\mathcal{U} = \{(x,y,z)/|\alpha| \le \tfrac{\pi}{2},\ x < a < 0,\ M = \tfrac{x^2}{4\rho} \gg 1,\ \tfrac{|x|}{\rho} \gg 1\}$$

At this stage the same analysis can be applied to partial derivatives of the potential.

Particular case of constant source strengths

Hence the leading behaviors of the highly oscillatory parts of the contributions from a segment and a panel of constant source strengths to the potential and its gradient can be assessed as follows:

$$\tilde{L}(P) = e^{z/2}(\mathbf{L_0} \cdot (1 + O(1/M)) \cdot \exp[-Me^{-i\alpha}] + \exp[-M]) \times \begin{pmatrix} n_X(\mathcal{S}) & \text{for}\mathcal{S} \\ 1 & \text{for}\mathcal{P} \end{pmatrix}$$

where $\mathbf{L_0}$ is given by the table below for the different cases and quantities, this for P in $\mathcal{U}$-regions of origin located at P_0', P_0' being one of the vertices of $\mathcal{S}$ or $\mathcal{P}$ on the free-surface.

source	potential	X-derivative	Y or Z-derivative
point	$K\rho^{-1/2}$	$KM^{1/2}\rho^{-1}$	$KM\rho^{-3/2}$
$\mathcal{S}$ $\parallel$ to the X-axis	$KM^{-1/2}$	$K\rho^{-1/2}$	$KM^{1/2}\rho^{-1}$
$\mathcal{S}$ $\nparallel$ to the X-axis	$K\rho^{1/2}M^{-1}$	$KM^{-1/2}$	$K\rho^{-1/2}$
$\mathcal{P}$ $\parallel$ to the X-axis	$K\rho M^{-3/2}$	$K\rho^{1/2}M^{-1}$	$KM^{-1/2}$
$\mathcal{P}$ $\nparallel$ to the X-axis	$K\rho^{3/2}M^{-2}$	$K\rho M^{-3/2}$	$K\rho^{1/2}M^{-1}$

The coefficients K are functions of: σ, k_0, the length of $\mathcal{S}$, or the area of $\mathcal{P}$.
The contribution from a segment or a panel non-parallel to the X-axis is given by the corresponding result for P in a $\mathcal{U}$-region, $\mathcal{U}_0$, which shrinks as the segment or the panel becomes more and more parallel to the X-axis, and by a behavior close to the 'parallel to the X-axis'-case in a region of the form $\mathcal{U}_1 \setminus \mathcal{U}_0$. For the case of $\mathcal{S}$, K goes to 0 as $n_X(\mathcal{S})$.

• We first notice that the contributions to the potential from $\mathcal{S}$ and $\mathcal{P}$ are converging as $\rho \to 0$, which is not the case for a source point. It is then obvious that the usual "approximation" of representing a segment of the waterline (or a panel at the waterline) of constant source strength by its centroid or any other point is simply *wrong*!

• Whether the contributions from $\mathcal{S}$ and $\mathcal{P}$ are considered separately or not, does not change their singular or non-singular nature: there is *no cancellation of the singularities*. Of course this does not imply that interferences do not occur outside a $\mathcal{U}$-region.

• The results for $\mathcal{P}$ parallel to the X-axis can be applied to a strut under the thin-ship approximation. It then appears that the main contribution to the wave elevation is like $K\rho^{1/2}M^{-1}$ (for a parabolic strut, by example) unless the source strength is 0 at the bow and the stern (cusped bow and stern) for which the contribution will be like $K\rho M^{-3/2}$ or weaker, depending on σ. For these simple cases it is obvious that the wake of the potential flow contains waves of short wavelengths. One can expect that, when dealing with the Neumann-Kelvin formulation, the contribution will be stronger.

• We did not mention how σ is obtained...(The treatment of the integral equation satisfied by σ is more delicate as the influence function involves derivatives of φ_F.)

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Tuck: You point out difficulties in solving Neumann-Kelvin problems, and suggest extreme care. I would go further, and suggest that Neumann-Kelvin problems be not solved at all. As I have been pointing out for too long now, the Neumann-Kelvin problem is inconsistent, and what is happening at the free surface is an inevitable consequence of this inconsistency. If people persist in ignoring this matter, they deserve the consequences. Non-thin ships make non-linear waves, and if non-linear terms are retained near the free surface, surely there will be no need for "relief" from problems such as the rapidly varying non-physical waves demonstrated by the author.

Clarisse: Whether solving Neumann-Kelvin problems is legitimate or notis still, I think, a matter of personal belief, and will remain such until someone proves that this class of problems has, or does not have, solutions; or until solutions to the non-linear problems will render the consideration of Neumann-Kelvin problems useless. It is certain that the validity of using of an exact boundary condition on the body and a linearized condition on the free surface is highly questionable, but this is independent of the type of source used. Panel methods using Rankine sources, for example, face extreme difficulties when reducing the size of the panels near the waterline. In any case, I would not associate the rapidly varying waves to this particular combination of boundary conditions. These waves *are* in the Kelvin wave-source potential and will appear in any solution to thin-ship problems or"Kelvin" problems (moving pressure disturbance at the free surface), and they are also partially there in the physical phenomena. Extreme care is , I think, necessary if one wants to deal with the Kelvin wave source potential for any of these problems. It is the way the rapidly varying waves contribute in the Neumann-Kelvin formulation – through the line integral – that could be detrimental to the existence of a solution. But after all, perhaps the "Kelvin" boundary condition is simply inconsistent for any kind of free surface disturbance!

Tulin: Is there any simple rule which would tell what part of the spectrum is reliable in the case where discrete sources are used at the waterline in a Neumann-Kelvin treatment? Is the energy containing part of the spectrum likely to be affected?

Clarisse: You suggest applying a filter to the Kelvin source potential that would give a contribution close to the ones I show for a segment or a panel. I think the best filter is the analytical integration itself: the contribution from a segment (panel) depends on the position in space of the segment (panel) and its size in a manner too complex to be easily simplified. I am *a priori* opposed to the idea of filtering the Kelvin wave source potential. This has been done too many times in the past by *ad hoc* arguments (including ones that ruled out the consideration of the line integral) with the results we know! If we want to solve a Neumann-Kelvin problem (and I am not advocating for solving Neumann-Kelvin problems), we should be consistent in our approach and keep all the information contained in the Kelvin wave source potential. This is feasible for the potential although it requires a lot of work which may in the end turn out to have been wasted, but this is the price that we must pay. Of course it would be nice to be able to state that a certain part of the spectrum of the Kelvin potential can be disgarded when solving a Neumann-Kelvin problem, but reasoning from the potential of a source point is erroneous: if the highly oscillatory waves with increasing amplitude which are found in the Kelvin potential are non-physical, so is the disturbance that generates them! I would only consider filtering the spectrum when robust solutions to the, mathematical, problem are available. It will then be possible to compare them with experiments and to assess which part of the spectrum is indeed relevant to the problem.