Trapping Modes Above Non-Cylindrical Bodies

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Using the theory of bounded symmetric linear operators together with Kelvin's minimum-energy theorem of classical hydrodynamics, Ursell(1987) was able to show that trapping modes exist above any infinitely long submerged horizontal cylinder which is symmetric about a vertical plane. The proof of that theorem relied on the knowledge of the existence of a trapping mode above a circular cylinder of small radius [Ursell(1951)].

In the discussion section of the 1987 paper, Ursell suggests that the arguments may possibly be extended to non-cylindrical bodies, and in particular, to a sphere situated at the centre of the channel (equivalent to an infinite row of equally spaced spheres.)

Thus, the modified Helmholtz equation used in the cylindrical case, must now be replaced by the three dimensional Laplacian

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.$$

To find a non-trivial solution of the homogeneous boundary value problem for a submerged sphere in a canal of width 2L, we start with the three dimensional point singular potentials, derived in Thorne(1953), with singular terms of the form

$$\frac{P_n^{2m+1}(\cos\theta)}{r^{n+1}}\sin(2m+1)\psi.$$

Suppose that the canal walls are given by the planes $z = \pm L$; then, with the multipole placed at $(0,h,0)$, an infinite system of 'canal-image potentials' at

$(0,h,\pm 2jL)$, $j=1,2,\ldots$ will be generated. We denote the image potential centred on $(0,h,2jL)$ by $\phi_{n,j}^{2m+1}$, and consider the sum

$$\Phi_n^{2m+1} = \sum_{j=-\infty}^{\infty} [\phi_{n,2j}^{2m+1} - \phi_{n,2j+1}^{2m+1}].$$

Then it follows by construction that Φ_n^{2m+1} has no flux across the canal walls. Also, it can be shown that if $K < \pi/2L$ $(= \beta_0$, say) then no waves propogate to infinity. The expression for Φ_n^{2m+1} has a series expansion of the form

$$\Phi_n^{2m+1} = \frac{P_n^{2m+1}(\cos\theta)}{r^{n+1}}\sin(2m+1)\psi + \sum_{s=1}^{\infty}\sum_{p=0}^{\bar{s}} B_{n;2m+1}^{s;2p+1}\, r^s P_s^{2p+1}(\cos\theta)\sin(2p+1)\psi$$

$(\bar{s} = s/2, (s-1)/2$ whichever is integral)

valid for $0 < a < \min\{2h, 2L\}$. We now assume that the velocity potential, ϕ, may be written as

$$\phi = \sum_{n=1}^{\infty}\sum_{m=0}^{\bar{n}} \Phi_n^{2m+1} \frac{a^{n+2} c_n^{2m+1}}{(n+1)\ (2m+1)!}$$

The unknown coefficients c_n^{2m+1} are determined from the body boundary condition and satisfy

$$c_n^{2m+1} - s(2p+1)! \sum_{n=1}^{\infty}\sum_{m=0}^{\bar{n}} \frac{B_{n;2m+1}^{s,2p+1} c_n^{2m+1}}{(n+1)\ (2m+1)!} a^{n+s+1} = 0$$

$(s=1,2,\ldots, p=0,\ldots,\bar{s})$

where the quadruple series can be shown to satisfy

$$\sum_{s=1}^{\infty}\sum_{p=0}^{\bar{s}}\sum_{n=1}^{\infty}\sum_{m=0}^{\bar{n}} \left| \frac{B_{n;2m+1}^{s,2p+1} a^{n+s+1}}{(n+1)\ (2m+1)!} s(2p+1)! \right| < \infty$$

if $0 < a < \min\{L, h/3, \frac{L(1-\exp\{-2h\pi/L\})}{3\pi(1+\exp\{-2h\pi/L\})}\}$; the convergence being uniform if

$$0 < \cot\gamma \le M(a\beta_0)^{-3} \qquad \text{where} \qquad K = \beta_0\cos\gamma$$

To show that the determinant of the above system of equations vanishes for some value of K,a,h,and L, it is possible to use Ursell's method of letting the radius of the sphere tend to zero whilst the wavenumber approaches β_0. See Ursell (1951).

The vanishing of the determinant implies the existence of a non trivial set $\{c_n^{2m+1}\}$ such that

$$\sum_{n=1}^{\infty}\sum_{m=0}^{\bar{n}}|c_n^{2m+1}| < \infty$$

and the corresponding velocity potential is a trapping mode. When the determinant vanishes an approximate relation between K and β_0 is given by

$$K \approx \beta_0(1-8(a\beta_0)^6\exp\{-4\beta_0 h\}+\ldots)$$

Using the fact that a trapping mode exists for a sphere of small radius, it should be possible to extend the work in Ursell(1987) and deduce the following theorem (c.f. Ursell,theorem 5.2)

Theorem: Suppose that S is any submerged closed body which is symmetrical about the central vertical plane of the canal. Then there exists at least one trapping mode.

This work is under investigation.

It was also stated that trapping modes can be proved to exist when the governing equation is the Helmholtz equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + k^2\phi = 0,$$

where the fluid is bounded internally by a rigid circular cylinder and bounded externally by two parallel rigid walls.

References

Thorne,R.C. 1953 Multipole expansions in the theory of surface waves. *Proc. Camb.Phil.Soc.* **49**,707-716.

Ursell,F. 1951 Trapping Modes in the Theory of surface waves. *Proc.Camb.Phil.Soc* 47,347-358.

Ursell,F. 1987 Mathematical aspects of trapping modes on the theory of surface waves. *J.Fluid.Mech.* 183,421-437.

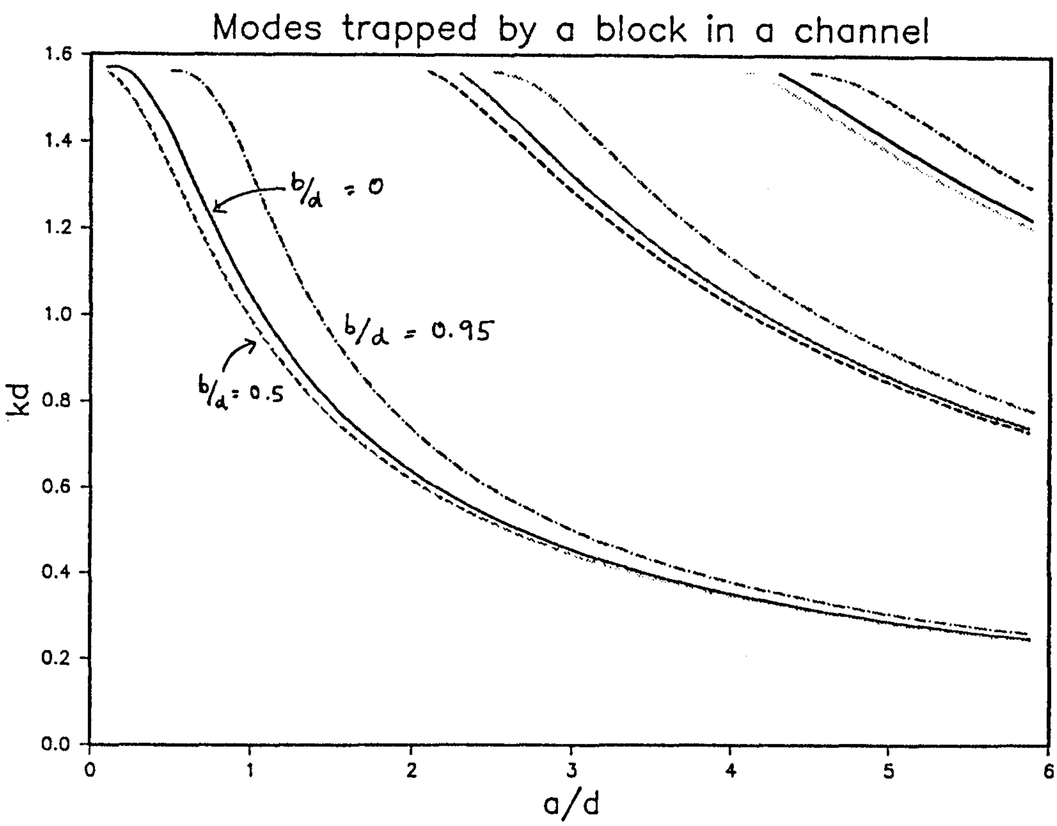
DISCUSSION

Evans: Mike Callan has stated that trapping modes can also be proved to exist for the Helmholtz equation when the fluid is bounded internally by a rigid circular cylinder and externally by two parallel rigid walls.

If, instead of a circular cylinder, we consider a cylinder of rectangular cross-section positioned symmetrically between the walls, eigenfunction expansions readily permit trapped-mode frequencies to be determined. Thus, let the walls be described by $|y| = d$, $-\infty < x < \infty$ and the sides of the rectangular cylinder by $|x| = a$, $|y| = b$, $0 \leq b < d$. Then, there exist trapped modes which continue to oscillate indefinitely in the vicinity of the cylinder and which vanish as $|x| \to \infty$. The problem can be regarded as a water-wave problem, in which the rectangular block is immersed throughout the entire water depth H and we are seeking discrete values of k. Then, the trapped-mode frequencies are given by ω where $\omega^2 = gk \tanh kH$. Alternatively, the problem can be regarded as arising in acoustics, in which case $\omega = kc$, where c is the velocity of sound.

Results for the variation of kd for trapped modes with a/d for different values of b/d are shown in the Figure. As $a/d \to 0$, $kd \to \pi/2$ the lowest cut-off frequency for the channel. Notice that varying b/d does not affect the modes significantly and that a flat plate on the centre-line (corresponding to $b/d = 0$) can support trapped modes. As a/d increases, further modes appear.

The discovering of these modes was prompted by a remark of Prof. Fritz Ursell during a visit to Bristol in February, 1990.



Newman: Is there a connection between these trapped waves and the resonant cross-waves near a wavemaker?

Callan: No. Approaching the cut-off frequency is a mathematical trick used to show the existence of trapping modes.

Peregrine: Are trapped modes likely to be associated with the higher cut-off frequencies?

Callan: The question is a very difficult one — is there a discrete spectrum embedded in the continuous spectrum? I have been working on this problem but achieved nothing.

Simon: Recent work by Weck [1] seems to indicate that your problem will be unique (i.e., there will be no trapping modes) for

$$\frac{h}{a} > \operatorname{cosec} 1 \simeq 1.188;$$

equally, this work would seem to rule out trapping modes for any sufficiently submerged body in a channel. However, Weck's extension of a correct 2-D result to 3-D is *incorrect*, and so uniqueness cannot be concluded from his work, in this situation.

Callan: That's a relief!

Reference

[1] N. Weck, 'On a boundary value problem in the theory of linear water waves', *Math. Methods in the Appl. Sciences* **12** (1990) 393–404.