CAUSALITY AND THE RADIATION CONDITION. II.

John V. Wehausen

Department of Naval Architecture and Offshore Engineering
University of California at Berkeley

The problem considered here is the same one treated in the First Workshop under the same title, that is, the motion of a floating body not under way subject to an external force and moment. We use the same notation as before: a right-handed coordinate system Oxyz with Oz upwards and Oxy in the equilibrium plane of the water; the "small" excursions about the equilibrium position of the body are denoted by $\alpha_1, \ldots, \alpha_6$, where the usual conventions about indices are observed. The linearized equations of motion can then be shown to have the following form (see the first report for references and further information about symbols):

$$(1) \qquad (m_{ik} + \mu_{ik})\ddot{\alpha}_k(t) + c_{ik}\alpha_k + \int_{-\infty}^{t} L_{ik}(t-\tau)\, \ddot{\alpha}_k(\tau)\, d\tau = X_i(t), \; i, k \in A,$$

where A is some subset of the integers $1, \ldots, 6$ indicating the allowable modes of motion (possibly all six, of course). Here the μ_{ik} are the added masses in Cummins' sense, the c_{ik} are the hydrostatic restoring forces and moments, and L_{ik} is a weighting function determined by solving an initial-value problem in potential theory. $L_{ik}(t)$ has the important property that it is 0 for $t<0$. We suppose, as before, that $X_i(t)$ is absolutely integrable so that Fourier transforms can be used in solving the equations of motion.

The solution leads to the paradoxical situation that future values of X_i are involved in determining the present value of $\alpha_i(t)$ unless it can be established that the determinant of

$$(2) \qquad \tilde{S}_{ik} = -\sigma^2 \int_0^{\infty} L_{ik}(t)\, e^{i\sigma t}\, dt + c_{ik} - \sigma^2[m_{ik} + \mu_{ik}(\infty)]$$
$$= -\sigma^2[m_{ik} + \mu_{ik}(\sigma)] + i\sigma\lambda_{ik}(\sigma) + c_{ik}, \quad i,k \in A,$$

has no zeros in the σ upper half-plane. In order to attack this problem the hermitian form Q associated with $\tilde{S}\bar{\tilde{S}}^T$ is introduced, for if this form is positive definite, then all the main-diagonal determinants of $\tilde{S}\bar{\tilde{S}}^T$ are positive and hence det $\tilde{S} \neq 0$. The proof of positive definiteness given at the First Workshop is valid only for real σ and not for all σ in the upper half-plane, as was pointed out to the author by Gyeong Joong Lee of Seoul National University. Consequently it is no proof at all. We attempt to correct this here.

By its definition $Q = \alpha_i \tilde{S}_{ij} \bar{\tilde{S}}_{kj} \bar{\alpha}_k \geq 0$. The assumption that Q=0 for some $\sigma = s+ir$, $r>0$, and some α_i, $i \epsilon A$, implies that the following equation must hold:

$$(3) \qquad \alpha_i c_{ik} \bar{\alpha}_k + \frac{2}{\pi} \int_0^\infty \alpha_i \lambda_{ik}(s') \bar{\alpha}_k \frac{(r^2+s^2)^2}{[r^2+(s+s')^2][r^2+(s-s')^2]} ds' = 0$$

($r>0$ has played a role in deriving this equation). One may show that

$$(4) \qquad \alpha_i c_{ik} \bar{\alpha}_k = \rho g W\{(\alpha_3+\alpha_4 y_c-\alpha_5 x_c)(\bar{\alpha}_3+\bar{\alpha}_4 y_c-\bar{\alpha}_5 x_c) + \\ + (x_c y_c - J_{12}/W)(\alpha_4\bar{\alpha}_5+\bar{\alpha}_4\alpha_5)\} + \rho g V\{H_1\alpha_5\bar{\alpha}_5 + H_2\alpha_4\bar{\alpha}_4\},$$

where (x_c, y_c) is the centroid of the waterplane area W, $J_{12}=\int_W xy\, dS$, H_1 and H_2 are the metacentric heights about the axes Ox and Oy, respectively, and V is the displaced volume. For a hydrostatically stable floating body the second line in (4) must be >0 if α_4 or α_5 is $\neq 0$. Hence $\alpha_i c_{ik} \bar{\alpha}_k \geq 0$. We may then conclude that (3) cannot be satisfied for any σ in the upper half-plane if $\alpha_i \lambda_{ik} \bar{\alpha}_k > 0$ or even if only $\alpha_i \lambda_{ik}(s') \bar{\alpha}_k \geq 0$ but > 0 for some interval of s'.

Although fluid dynamics has provided the definitions of $\mu_{ik}(\infty)$ and $L_{ik}(t)$ and their symmetry in i and k, the remainder of the reasoning has involved only the form of the equation (1) and properties of c_{ik}.

We have not yet been able to prove that the positive definiteness of Q implies that $\alpha_i \lambda_{ik} \bar{\alpha}_k$ is positive definite. Also, in the discussion at the First Workshop it was stated that if the body is under way causality is implied by the positive definiteness of $x_i[\sigma(\lambda_{ik}+\lambda_{ki})+i\sigma^2(\mu_{ik}-\mu_{ki})]\bar{x}_k$. This is contradicted by the known existence of bodies and positive Froude numbers for which, say, $\lambda_{33}<0$. The problem is still open.

Workshop on Water Waves and Floating Bodies

Discusser: Enok Palm

At the end of your talk you wanted to solve the problem as an initial-value problem and then obtain that y ou do not need any radiation conditions.

I believe that should be possible, but I also believe that you must use some boundary conditions at infinity for all time, expressing that energy there is not infinite. That could perhaps be obtained by requiring that the solution can be obtained as a Fourier integral.

Author's reply: J. V. Wehausen

What I would like to prove is that $Q>0$ everywhere in the σ upper half-plane implies that $\alpha_i \lambda_{ik}(s) \bar{\alpha}_k \geq 0$ for all s and is > 0 for some interval. It isn't clear to me that this is the same as an initial-value problem, but it certainly seems analogous to avoiding invoking a radiation condition by solving such a problem.

Discusser: E. O. Tuck

If $\alpha_i c_{ik} \bar{\alpha}_k < 0$, then the last contradiction does not apply, which is consistent with the Ursell comment about exponentially increasing solutions, since that would lead to instability.

Author's reply: J. V. Wehausen

If $\alpha_i c_{ik} \bar{\alpha}_k < 0$, then I suppose that (3) can be satisfied for some choice of $r>0$ and s. However, although $Q=0$ implies (3), I am not able to prove the converse.

F. Ursell. Discussion of Wehausen's paper.

It is interesting to approach the problem from a slightly different point of view. For the sake of definiteness let us consider the transient motion of a half-immersed circle $r = a$, $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$ subject to an applied force, see Ursell 1964. This motion is symmetric about $\theta = 0$.

Transient problems are usually solved by the method of Laplace transforms, let $\varphi(x,y,t)$ denote the velocity potential, suppose that $|\varphi(x,y,t)| < Me^{ct}$, and write $\Phi(x,y,\omega) = \int_0^\infty \varphi(x,y,t)e^{i\omega t}\,dt$ where $\operatorname{Im}\omega > c$, then $|\Phi(x,y,\omega)| < M/(\operatorname{Im}\omega - c)$.

(Φ is the conventional Laplace transform $\int_0^\infty \varphi(x,y,t)e^{-pt}\,dt$, with $p = -i\omega$.) Then we have

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\Phi(x,y,\omega) = 0 \quad \text{in the fluid,}$$

and $\omega^2\Phi + g\dfrac{\partial\Phi}{\partial y} = 0$ on the mean free surface $y = 0$, (with two other boundary conditions on the body). These are the usual equations for a periodic motion with angular frequency ω, except that here ω is complex and there is no radiation condition. Should a radiation condition be imposed? (This question seems to be related to Wehausen's discussion of causality.) Let us find the form of the expansion at large distances, we shall see that no radiation condition is needed. We define the wave source potential

$$S(x,y,\omega) = \int_0^\infty \frac{e^{-ky}\cos kx}{k - \frac{\omega^2}{g}}\,dk,$$

there is no singularity on the real k-axis since ω is complex. Following the line of argument of Ursell 1950 or Ursell 1968 we find that

$$\Phi(x,y,\omega) = A(\omega)\,S(x,y,\omega) + F_1(x,y,\omega) + F_2(x,y,\omega)$$

where $\quad F_1(x,y,\omega) = \sum_{1}^{\infty} \alpha_m(\omega) \cos m\theta . r^{-m}, \quad r > a,$

and $\quad F_2(x,y,\omega) = \sum_{0}^{\infty} \beta_m(\omega) \cos m\theta . r^{m}, \quad r > a.$

In fact, since $F_2(x,y,\omega)$ is a series of positive powers, it must be convergent for $0 \leqslant r < \infty$.

For real ω it is known that $F_2(x,y,\omega)$ is a standing wave determined by the radiation condition, but we shall see that $F_2(x,y,\omega) = 0$ for complex ω. We have just noted that $F_2(x,y,\omega)$ is defined for all (x,y), even inside the body, and that

$$\left[\frac{\omega^2}{g} + \frac{\partial}{\partial y}\right] F_2(x,y,\omega) = \left[\frac{\omega^2}{g} + \frac{\partial}{\partial y}\right] \Phi(x,y,\omega) - A(\omega)\frac{y}{x^2+y^2} - \left[\frac{\omega^2}{g} + \frac{\partial}{\partial y}\right] F_1(x,y,\omega),$$

where the functions on the right are defined when $x^2 + y^2 > a^2$, both when $y > 0$ and when $y < 0$ since they satsify the free-surface condition when $y = 0$. Also these functions are bounded when $x^2 + y^2 \to \infty$ and vanish when $y = 0$. It follows that $\left[\frac{\omega^2}{g} + \frac{\partial}{\partial y}\right] F_2(x,y,\omega) = 0$, and since also $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] F_2(x,y,\omega) = 0$, we conclude that

$$F_2(x,y,\omega) = C(\omega) \exp\left[-\frac{\omega^2}{g} y\right] \cos \frac{\omega^2}{g} x,$$

as for real ω; but for complex ω this is not consistent with the condition

$$|\Phi(x,y,\omega)| < \frac{M}{\text{Im } \omega - c}$$

unless $C(\omega) = 0$, since the other terms in the expansion of $\Phi(x,y,\omega)$ tend to 0 when $x \longrightarrow \infty$ while $\cos \frac{\omega^2}{g} x \longrightarrow \infty$ for complex ω, thus $F_2(x,y,\omega) = 0$. We also note when $\omega \to$ a real positive value ω_o, the source potential $S(x,y,\omega) \longrightarrow \int_0^{\infty} \frac{e^{-ky}\cos kx}{k - \frac{\omega_0^2}{g}} dk,$

with an indentation below ω_0. (This is the usual radiation condition.) We can now apply the inversion formula

$$\varphi(x,y,t) = \frac{1}{2\pi} \int_{ic-\infty}^{ic+\infty} \Phi(x,y,\omega)\, e^{-i\omega t}\, d\omega.$$

If $\Phi(x,y,\omega)$ is sufficiently small when $\operatorname{Im}\omega \to \infty$ (and it usually is) we find that $\varphi(x,y,t) = 0$ when $t < 0$, and that $\varphi(x,y,t)$ involves the applied force up to time t only; see Ursell 1964, where the displacement at time t is given for the semicircle. So there is no difficulty with causality when we use the Laplace transforms.

References

Ursell, F. 1950 Surface waves on deep water in the presence of a submerged circular cylinder II. Proc. Camb. Phil. Soc. 46, 153-158.

Ursell, F. 1964 The decay of the motion of a floating body. Jl. Fluid Mech. 19, 305-319.

Ursell, F. 1968 The expansion of water-wave potentials at great distances. Proc. Camb. Phil. Soc. 64, 811-826.

Wehausen: Although Ursell's discussion of the initial-value problem by use of the Laplace transform is of much interest of itself, I don't believe that it clarifies very much the problem that I am trying to treat. In the first place, I am not really treating an initial-value problem although I think that I may have given that impression. The forcing function $\gamma_1(t)$ is given for $-\infty<t<+\infty$ it is also required to be absolutely integrable, but this is so that I can use ordinary Fourier transforms (instead of, say, Wiener's generalized harmonic analysis). What seems like a straightforward use of Fourier transforms to solve the equations of motion has led to the paradoxical situation that $\alpha_j(t)$ is determined in terms of $x_k(t+\tau)$ for $\tau>0$ as well at $\tau<0$ unless a certain weighting function $T_{ik}(t)$ can be shown to vanish for $t<0$. If I have not made (another) egregious error, this will be true for a hydrostatically stable floating body if $\alpha_i\lambda_{ik}(S)\alpha_k\geq 0$ for all s and >0 for at least some interval of s. This is not, of course, the usual statement of the radiation condition, but related to it. Also, "causality" gives to the title a philosophical aspect that is belied by the content of the paper. Succinctness is my only defense for this.