The Scattering of Waves by Vertical Cylinders

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Introduction

The problem of the scattering of waves by a group of N vertical cylinders was considered by Spring and Monkmeyer (1974) who showed how, in principle, the velocity potential could be calculated for a wave incident upon them. However the actual calculations were performed for two cylinders only. Eatock Taylor and Hung (1985) looked at the problem from the point of view of drift forces but again only considered the special case of two cylinders. McIver and Evans (1984) developed an approximate method for calculating the potential and showed comparisons with the method of Spring and Monkmeyer for groups of cylinders with $N \geq 2$. The theory for the N cylinder case is not stated however.

The purpose of this paper is to show that the solution to the N cylinder case can be expressed in a form which is only marginally more complicated than the solution to the straightforward one cylinder case. All the previous work mentioned above has concerned the calculation of integrated quantities associated with the cylinders such as first and second order forces. Here we will also be concerned with the calculation of free surface amplitudes over a region containing the N cylinders. This involves being able to accurately calculate the potential.

Formulation

In order to simplify the analysis let the velocity potential

$$\Phi(x,y,z,t) = \mathrm{Re}\{\phi(x,y)f(z)e^{-i\omega t}\}$$

where

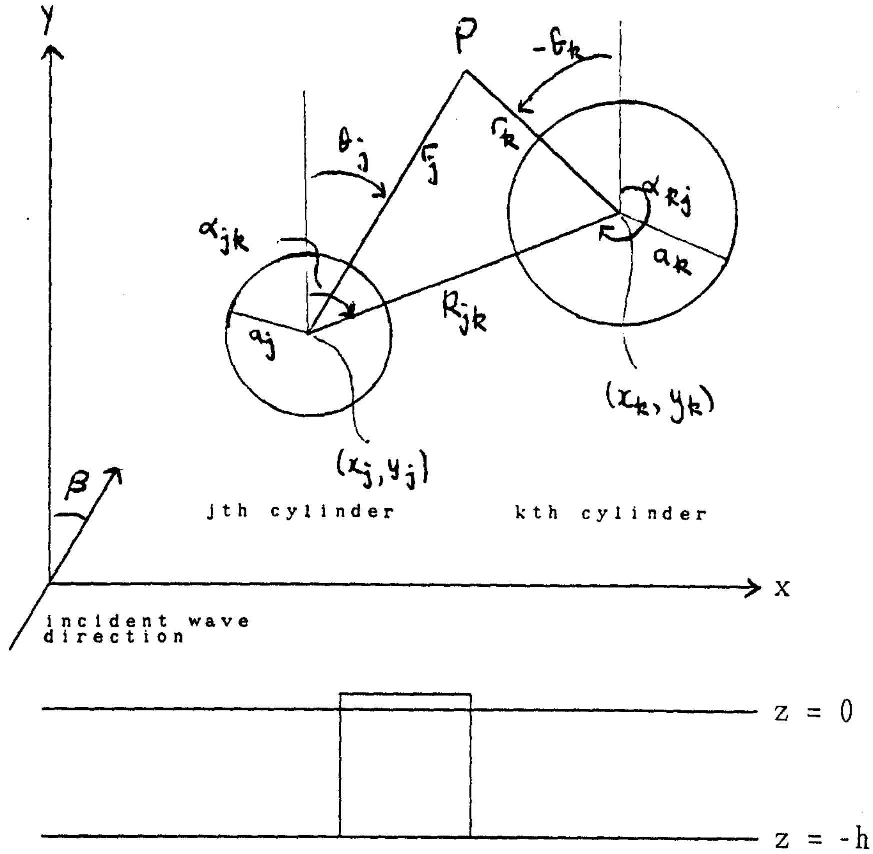
$$f(z) = -\frac{igA}{\omega}\frac{\cosh \kappa(z+h)}{\cosh \kappa h}, \quad A \text{ real},$$

and

$$\kappa \tanh \kappa h = K \equiv \omega^2/g.$$

Plan view of
two arbitrary
cylinders

Elevation view
of one cylinder



Definition Sketch

With this choice the time independent free surface elevation is given by

$$\eta(x,y) = A\phi(x,y).$$

We will use N+1 coordinate systems : (r,θ) will be polar coordinates centered o the origin and (r_j,θ_j) will be polar coordinates centred on (x_j,y_j), the centre o the jth cylinder. In all cases θ will be measured clockwise from the positiv y-axis.

It can be shown that the total velocity potential can be expressed in the form

$$\phi = \exp(i\kappa r\cos(\theta-\beta)) + \sum_{j=1}^{N} \sum_{n=-\infty}^{\infty} A_n^j\, Z_n^j\, H_n(\kappa r_j)\, \exp(-in\theta_j) \tag{1}$$

for some set of complex numbers A_n^j, where

$$Z_n^j = J_n'(\kappa a_j)/H_n'(\kappa a_j)\ .$$

Applying the boundary conditions on each of the cylinders leads to an infinite syste of equations which we truncate to give the N(2M+1) equations

$$A_m^k + \sum_{\substack{j=1\\ j\neq k}}^{N} \sum_{n=-M}^{M} A_n^j\, Z_n^j\, \exp(-i(n-m)\alpha_{jk})\, H_{n-m}(\kappa R_{jk}) = -\, I_k\, \exp(im(\tfrac{\pi}{2}+\beta)) \tag{2}$$

$$k=1,\ldots,N\ ;\ m=-M,\ldots,M$$

which can be solved for the unknown coefficients A_m^k. Here

$$I_k \equiv \exp(i\kappa(x_k \sin\beta + y_k \cos\beta))$$

is a phase factor associated with the kth cylinder.

Using addition theorems for Bessel functions (1) can be expressed solely in terms of the coordinates (r_j, θ_j). Substituting from (2) into the resulting expression for $\phi(r_j, \theta_j)$ gives, after some simplification,

$$\phi(r_j,\theta_j) = \sum_{n=-\infty}^{\infty} A_n^j \, (Z_n^j \, H_n(\kappa r_j) - J_n(\kappa r_j)) \exp(-in\theta_j) \tag{3}$$

provided $\qquad r_j < R_{jk} \quad \forall \, k \, .$

In particular, using the Wronskian relation for Bessel functions,

$$\phi(a_j,\theta_j) = -\frac{2i}{\pi\kappa a_j} \sum_{n=-\infty}^{\infty} \frac{A_n^j}{H_n'(\kappa a_j)} \exp(-in\theta_j) \; . \tag{4}$$

The time-independent amplitude of the first order force on the jth cylinder can then be calculated straightforwardly. We get

$$|X^j| = \tfrac{1}{2} \, |F| \, |A_{-1}^j \begin{Bmatrix} - \\ + \end{Bmatrix} A_1^j|$$

where the upper elements of the bracketed pairs refer to the force in the y direction and the lower ones to force in the x direction.

$$F \equiv \frac{4\rho g A}{\kappa^2} \frac{\tanh \kappa h}{H_1'(\kappa a_j)}$$

is the force the cylinder would experience in the direction of the incident wave if it were in isolation.

The drift force on the jth cylinder can also be calculated using (4) and the resulting formula is again just a single sum involving the coefficients for that particular cylinder.

In order to calculate the free surface amplitude outside the cylinder group (3) is insufficient. Instead we must revert to (1) which for large groups of cylinders is a disadvantage requiring approximately N evaluations of functions like $\sum_{n=-\infty}^{\infty} A_n \, H_n(\kappa r) \exp(-in\theta)$ when N is large. Reference to McIver and Evans (1984) shows that this is still much better than using their approximate method which requires approximately N^2 such evaluations when N is large.

Results showing the various quantities discussed above together with the relati merits of using the exact theory over the approximate theory of McIver and Evans wi be shown.

Conclusion

The exact theory for the scattering of waves by N vertical cylinders was propos by Spring and Monkmeyer (1974) but has never been fully exploited. Majo simplifications can be made to the resulting formulas which make the numerica calculation of quantities like the first and second order forces on the cylinders a simple and efficient as it is for a single cylinder. Far away from the cylinder these simplifications are not available but when calculating free surface amplitude the method still has many advantages over the approximate theory of McIver and Evan (1984) particularly when N is large.

Acknowledgement

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DISCUSSION

Eatock Taylor: These results are extremely fascinating. The Kochin function argument shows that similar phenomena to the N^2 drift force ratio arise in the case of hydrodynamic damping. It might be interesting to examine the *damping* contribution from one cylinder in a multi-cylinder group - could one exploit the interaction effects in a positive design approach (e.g. independent floating bodies)?

Linton & Evans: It may well be possible to extend the ideas shown here to cover the case of hydrodynamic damping and this is something that will be looked into in the future.

Kleinmann: I don't wish to accuse you of not doing something that no one else does either but have you verified that the solution of the truncated system converges to the solution of the infinite system. I believe there are tests for convergence of this so-called "abschnitz-methoden" and suggest that it would be worth while to check and see if they give positive answers.

Linton & Evans: I have not performed any analysis on the convergence of the truncated system and have simply relied on numerical observation and a comparison of my results with those of other authors using different methods. This may not be very satisfactory mathematically but I think that the additional insight gained by proving convergence may not be worth the time & effort required.

Ursell: Truncation: I would expect the truncated solution to converge to the exact solution. Recent work on the method of multipoles (for the heaving circular cylinder) suggests that the approach to exact solution is quite slow, and it would be interesting to investigate this in more detail.

Linton & Evans: This is also a reply to Kleinmann's question.

Greenhow: For calculation of the necessary air gap below a rig deck, long design waves ~300m are sometimes used. Does your theory differ from McIver and Evans more or less as the waves get long?

Linton & Evans: Preliminary results suggest that the plane wave approximation of McIver & Evans becomes more accurate in long waves which is perhaps surprising due to the nature of the approximation.