The wide-spacing approximation applied to multiple scattering and sloshing problems

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Introduction

At the first Workshop in 1986, McIver and Evans presented a paper on sloshing frequencies, in which they described a method for obtaining the resonant frequencies of oscillation of fluid in a two-dimensional rectangular tank containing a thin baffle. The method involved the use of eigenfunction expansions and the solution of an integral equation for the unknown velocity above or below the baffle.

An approximate eigenvalue relation (equation 19) was also given for an arbitrarily shaped baffle on the basis of the wide-spacing approximation in terms of the (assumed known) reflection and transmission coefficients for the baffle in an infinitely wide tank. It was shown that this approximation was remarkably good at predicting even the lowest eigenvalue where the wide-spacing assumption is clearly invalid.

We return to this approximation here but we use a different approach. The expression given by McIver and Evans for the determination of the resonant frequencies is unsatisfactory since it is complex, whereas the solutions sought must, on physical grounds, be real. We show, using the new approach, how an alternative equation can be derived, which is real and which, in the case of a symmetric obstacle, depends solely on the phases of the waves radiated to infinity due to forced symmetric and antisymmetric motions of the body. The method can be extended to any number of obstacles and it is shown how a closed form expression can be derived for the eigenvalue relation or for the reflection and transmission coefficients for the case of n identical equally-spaced obstacles.

Formulation and Solution

The tank is assumed to occupy $-c \leq x \leq b$, $0 \leq y \leq h$ so that $y = h$ is the undisturbed free surface with the obstacle positioned around $x = 0$. In the vicinity of the right hand wall $x = b$, assumed to be far removed from the obstacle we may write

$$\phi \sim \{B\, e^{ik(x-b)} + B\, e^{-ik(x-b)}\} \cosh ky \qquad (1)$$

satisfying the condition of zero normal velocity on $x = b$, whilst in the vicinity of the left hand wall $x = -c$, also assumed far removed from the obstacle we may similarly write

$$\phi \sim \{C\, e^{ik(x+c)} + C\, e^{-ik(x+c)}\} \cosh ky. \qquad (2)$$

Here k is the positive real root of the equation

$$\omega^2/g \equiv K = k \tanh kh. \qquad (3)$$

and is to be determined, whence the resonant frequency ω is given by (3).

Consider, in the infinite strip $-\infty < x < \infty$, $0 \leq y \leq h$, the radiation potential $\psi_i(x,y)$ whose normal velocity is real on the obstacle, perhaps corresponding to a rigid body motion, and which satisfies

$$\psi_i \sim A_i^+\, e^{ikx} \cosh ky, \quad x \longrightarrow +\infty$$

$$\sim A_i^-\, e^{-ikx} \cosh ky, \quad x \longrightarrow -\infty$$

Define $$\chi_i \equiv \psi_i - \bar{\psi}_i$$

where a bar denotes complex conjugate. Then the normal derivative of χ_i vanishes on the obstacle. Under the wide-spacing approximation we shall assume that (1), (2) hold as $x \longrightarrow +\infty, -\infty$ respectively.

Then application of the identity

$$\int_C (\phi \frac{\partial \chi_i}{\partial n} - \chi_i \frac{\partial \phi}{\partial n}) ds = 0$$

around all boundaries closed by lines $x = \pm X$, $0 \leq y \leq h$, where X is large enough for the asymptotic forms of ϕ, χ_i to hold, gives

$$|A_i^-| C \cos(kc + \theta_i^-) + |A_i^-| B \cos(kb + \theta_i^+) = 0 \qquad (3)$$

where $A_i^{\pm} = |A_i^{\pm}| e^{i\theta_i^{\pm}}$.

By choosing first $i = 1$ and then $i = 2$ corresponding to a different independent rigid body motion, we obtain two homogeneous equations for B, C.

In the interesting case of a body symmetric about $x = 0$ for which $A_1^+ = A_1^- \equiv A_s$, $\theta_1^+ = \theta_1^- \equiv \theta_s$, $A_2^+ = -A_2^- \equiv A_a$, $\theta_2^+ = \theta_2^- \pm\pi \equiv \theta_a$, we obtain

$$\cos(kc+\theta_s)\cos(kb+\theta_a) + \cos(kb+\theta_s)\cos(kc+\theta_a) = 0. \tag{4}$$

This is a real equation from which, given b, c, θ_a, θ_s, the wavenumber k and hence the resonant frequencies can be determined. Now the Newman relations for a symmetric body are

$$R + T = -e^{2i\theta_s}, \quad R - T = -e^{2i\theta_a}. \tag{5}$$

It is known that for a thin baffle $R + T = 1$ or $\theta_s = \frac{\pi}{2}$. It then follows from (5) that $\tan\theta_a = iR/T$.

Substitution of these results into (4) and re-arrangement gives

$$\sin ka = 2iRT^{-1}\sin kb \sin kc, \quad (a = b+c)$$

in agreement with McIver and Evans.

Again, for a symmetrically-positioned symmetric obstacle, $b = c$ so (4) reduces to

$$\cos(kb+\theta_s)\cos(kb+\theta_a) = 0 \tag{6}$$

with solutions
$$kb = \begin{cases} -\theta_s + (2n-1)\pi/2, & n = 1,2,\ldots \\ -\theta_a + (2m-1)\pi/2, & m = 1,2,\ldots \end{cases} \tag{7}$$

This is confirmed by the McIver and Evans result which reduces to $R \pm T = e^{-2ikb}$ in this case, and which, after use of (5) agrees with (7).

Results based on (4) will be presented and compared with results for a symmetrically positioned rectangular block on the floor of the tank obtained by solving the full linear problem using matched eigenfunction expansions and integral equations (Watson, private communication).

The above ideas can be extended to treat any number of obstacles. If the obstacles are identical and equally spaced, considerable simplification occurs and explicit results obtained for both the reflection coefficient in the scattering problem, and the eigenvalue relation to determine the resonant frequencies.

We consider n identical symmetric bodies, each contained in an interval of length a, its centre a distance b from one end, c from the other. At the left boundary of the m^{th} interval we assume travelling waves of amplitude A_m, B_m and at the right boundary, A_{m+1}, B_{m+1}. The ideas described above provide a matrix relation between these quantities in the form

$$\begin{pmatrix} -e^{i(kc+\theta_s)} & -e^{-i(kc+\theta_s)} \\ e^{i(kc+\theta_a)} & e^{-i(kc+\theta_a)} \end{pmatrix} \begin{pmatrix} A_m \\ B_m \end{pmatrix} = \begin{pmatrix} e^{-i(kb+\theta_s)} & e^{i(kb+\theta_s)} \\ e^{-i(kb+\theta_a)} & e^{i(kb+\theta_a)} \end{pmatrix} \begin{pmatrix} A_{m+1} \\ B_{m+1} \end{pmatrix} \qquad (8)$$

for $m = 0,1,2,\ldots n$.

The special case $m = 0$, $A_0 = B_0$, $A_1 = B_1$ corresponding to the sloshing problem for a single body confined between walls, can be seen from (8) to reduce to (4).

It will be shown, by successive application of (8), how A_n, B_n can be expressed in terms of A_0, B_0 in closed form and hence how an explicit form can be obtained for the reflection and transmission coefficients, as well as the conditions for resonance in this more general case.

DISCUSSION

Newman: In the figures showing kb as a function of h for a rectangular bottom obstacle it would seem that kb should tend to non-zero (standing wave) values in both limits $h \to 0$ and $h \to 1$.

Evans: Theorems suggest that as you reduce the water volume by increasing the size of the obstacle, for the same free surface the sloshing frequency showed decrease monotonically. It is not entirely clear that this applies here in the limit as the block breaks the surface. For the geometry shown one might expect the first few curves to tend to zero as the height of the block increases. I am unable to explain as yet why curves also appear to tend to zero.

Mehlum: You showed slides with remarkable agreement between "exact theory" and your wide spacing approximation for square boxes. Have you tried a similar comparison for submerged cylinders?

Evans: We have obtained approximations to sloshing frequencies in a tank containing a submerged cylinder but have not yet developed an 'exact' theory with which to compare our results.