Numerical Investigation on the Desingularization of Boundary Integral Equation for the Three Dimensional Nonlinear Wave Problems

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The mixed Eulerian-Lagrangian method, originated by Longuet-Higgins and Cokelet (1976), is very powerful for solving time dependent nonlinear free surface problems. The method is a time marching procedure which requires two major computations at each time step: (1) solve a boundary value problem (BVP); (2) update the free surface position and the potential on it by integrating the nonlinear free surface conditions. Since most of the computation time will be spent on the BVP, it is important to have an effective solution method for the time marching procedure to be of practical use.

The boundary integral method (BIM) is a powerful alternative for the solution of the BVP. There are two versions of BIM: the direct and indirect versions. In the direct version, the boundary integral equation (BIE) is derived from Green's second identity and the solution is obtained directly by solving the BIE. In the indirect version, the solution is assumed to be a superposition of some sort of singularity distribution over some region. By applying the boundary conditions, one obtains a BIE for the unknown strength of the singularity. Usually, the singular points are located on the boundary surfaces and singular formulations of the BIE are obtained. These formulations have been studied and used extensively. However, there are difficulties in the evaluation of surface integrals with singular kernels.

If the singular points are placed away from the boundary surfaces and outside the fluid domain, non-singular formulations of the BIE are obtained. By applying Green's theorem to ϕ and a simple wave source with the singular point lying inside the body (i.e. outside the fluid domain) and using a bilinear expansion of the source, one can obtain the so called "null-field equations for the water wave radiation problems" with unique solutions (Martin, 1981). Heise (1978) studies some numerical properties of integral equations in which the singular points are on an auxiliary boundary outside the solution domain for plane elastostatic problems. Han and Olson (1987) use an adaptive method in which the singularities are located outside the domain of the problem and allowed to move as part of the solution process. Jensen, Mi and Söding (1986) solve the wave resistance problem successfully using a Rankine source method in which the sources are distributed above the free surface. All of these works show the effectiveness of non-singular formulations.

The following are several expected advantages of the desingularization over the singular formulations, some of which have been studied by previous investigators:

- Since the kernels are non-singular and special care is not required for the singular contribution, the integrals can be performed by simple numerical quadratures.
- Fewer nodes may be required since simple quadratures ease the restrictions of a flat panel.
- There is a larger flexibility of choosing the fundamental solutions. Higher order singularities can be used more easily.
- An overdetermined system can be easily obtained, which may lead to better results. The calculations of Schultz and Hong (1988) show the effectiveness of a desingularized and overdetermined system in two dimensions. Their results show greater accuracy and faster convergence.

Compared with the direct version, the indirect version has two more advantages:

- Integrals of singularity distributions may be replaced by a summation of concentrated singularities, which makes the computation even simpler.
- Since the indirect version requires no integration over the far field enclosure, one may expect smaller error due to the trunction of the free surface and fewer unknowns may be required.

However, the desingularization also causes some problems, which may lead to incorrect solutions if treated improperly: (1) little experience on the placement of the singular points is available, and (2) uniqueness and completeness have not been proved for general problems. However, if the singular point is located a distance away from the boundary surface in proportion to the local mesh size, and as the mesh becomes finer, the singular point will get closer to the the surface. In the limit, our non-singular formulation becomes identical to the singular formulation.

In this paper, the influence of the distance of the singular points off the surface and the convergence of the non-singular formulations are studied numerically with some simple potential problems. The results of a three dimensional flow consisting of a dipole under a flat, $\phi = 0$, free surface are shown in Figures 1 and 2. The distance is characterized by $Z_a = Z_h/\sqrt{A}$. Z_h is the dimensional distance and A is the average area of the local adjacent meshes. The computed values of the errors of the normal derivatives $\frac{\partial \phi}{\partial n}$ on the free surface are measured by $E_2 = \frac{1}{M}\sqrt{\sum \varepsilon_i^2}$, where $\varepsilon_i = (\frac{\partial \phi}{\partial n})_{num} - (\frac{\partial \phi}{\partial n})_{exact}$ is the error at each nodal point and M the number of nodal points.

Figure 1 shows the results of the the direct version. The results with the exact evaluation of the panel integration using Newman's approach (1986) are also plotted with dotted lines for comparison. As expected, for small Z_a (< 0.25), the results using 2x2 Gauss quadrature have larger errors since Gauss quadrature is inaccurate for small Z_a. For large Z_a, the solid lines merge to the dotted lines since Gauss quadrature is accurate. However, the solution is deficient because the singular points are too far away from the surface to capture the nearly singular behaviors of the system. It is surprising that in the middle range of Z_a, the results using Gauss quadrature show less error than those using Newman's approach. The reason is not quite clear at this moment. One possibility is that the errors from different sources (for example, discretization and numerical quadrature) cancel each other. Figure 2 shows the results of of the indirect version. The solution is assumed to be the sum of concentrated sources above the free surface. As can be seen, the errors are too large for the solutions to be acceptable for small Z_a because concentrated sources can not model the flow well when they are too close to the surface in our formulation. However, for quite a wide range of Z_a, $0.5 < Z_a < 3.0$, the solutions are very good.

The computation time required for the matrix set up by 2x2 Gauss quadrature is about 65% less than that required by Newman's approach, and the time required for the indirect version is about a quarter of that required for 2x2 Gauss quadrature. The algebraic system is less well conditioned as Z_a gets larger so more iterations are needed. The overdetermination of the system with the direct version provides no benefit.

Figures 3 and 4 show calculations of waves generated by a source and sink of unit strength moving horizontally 1.5 units under a free surface. The distance between the source and the sink is unity and the Froude number based on the depth of the submergence is 1.738. The nonlinear results updating the free surface using the Euler scheme are compared with the linear results of unsteady Kelvin sources with the time convolution integrals evaluated numerically. The comparisons are qualitatively good although there are some differences yet to be analysed.

Additional studies will investigate the use of higher order singularities, overdetermination of the system with the indirect version, and the use of higher order time integration schemes.

Acknowledgement

This work was supported under the Program in Ship Hydrodynamics at The University of Michigan, funded by The University Research Initiative of the Office of Naval Research, Contract No. N000184-86-K-0684.

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Martin: Regular integral equations are regularly rediscovered. They have been used in potential theory, acoustics and elasticity, and their properties have been studied; see e.g., the work of S.Christiansen and of U.Heise.

Have you examined the conditon number of your systems, as a function of Z_a? I think that you will find that the condition number will be least when the points P are located on the boundary S!

Cao, Schultz & Beck: We agree that the condition number of the system is minimized when the singular points are located on the boundary. However, the condition number does not necessarily have a direct relationship with accuracy of the BIM discretization as shown in Figures 1 and 2. In addition, our conjugate gradient matrix solver does not appear to be sensitive to the condition number although the number of iterations becomes large as Z_a increases. When we desingularize on the order of the mesh size, the condition number appears to be adequate.

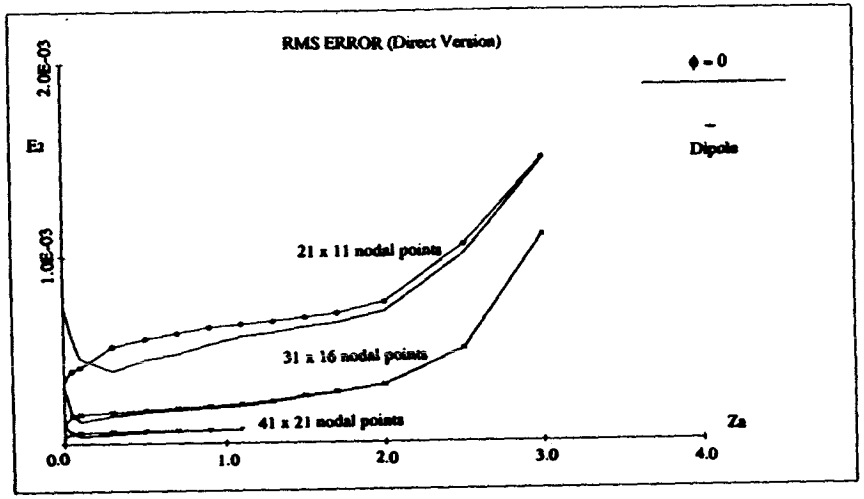

Fig. 1
RMS Error of Normal Derivative
(Direct Version)

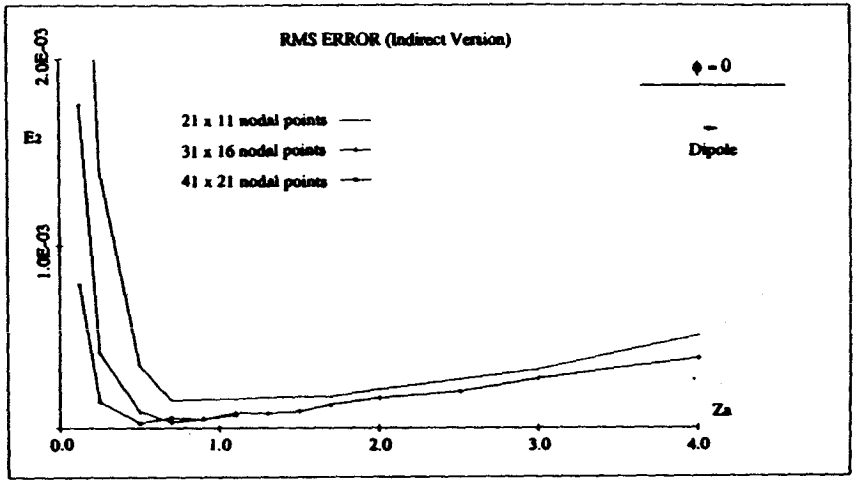

Fig. 2
RMS Error of Normal Derivative
(Indirect Version)

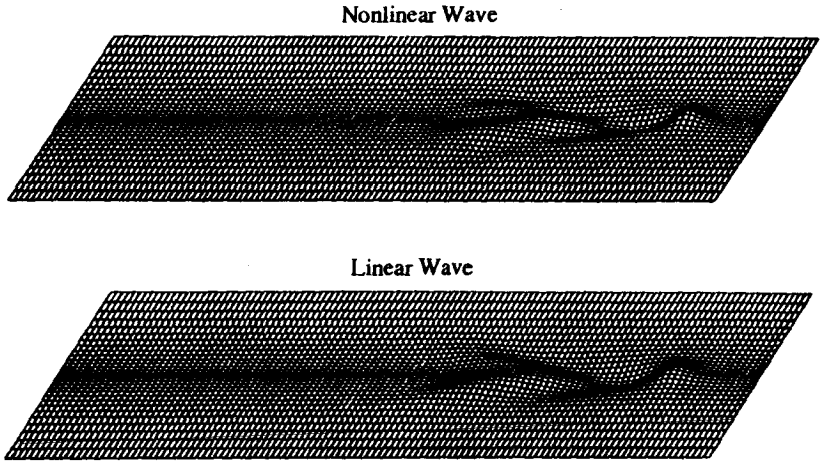

Fig. 3
Wave Patterns at T = 2.5

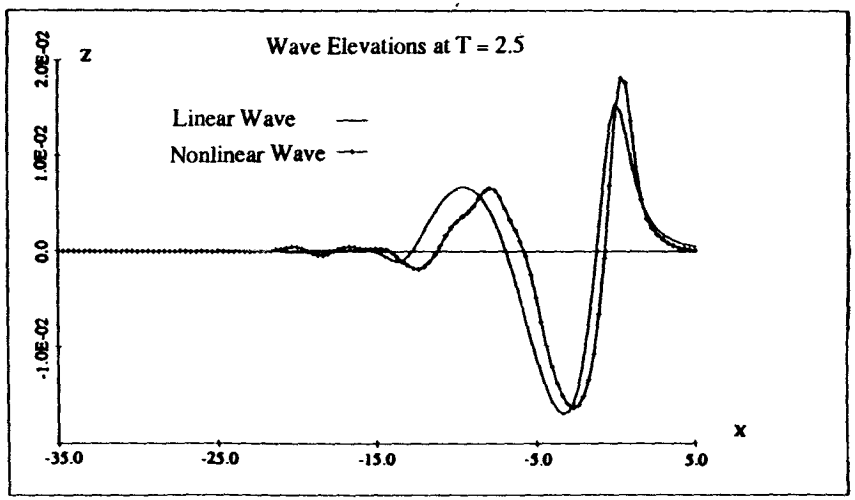

Fig. 4
Wave Elevations along the Plane of Symmetry
at T = 2.5

DISCUSSION

Bertram: 1. Why do you use time-stepping to solve steady problems? 2. I suggest enforcing a corresponding linear condition on the free surface in your time-stepping instead to have an accurate (as possible) comparison. Later you can go to non-linear conditions and see what changes. Even for small non-linearities you introduce unnecessary errors.

Cao, Schultz & Beck: 1.We are not merely looking at the steady problem. Another reason for using time marching procedure is to delay the difficulties of the radiation condition.
2. We have already done the linear calculations. The results suggest that a more accurate time integration or spatial resolution is necessary.

Eatock Taylor: It is interesting to see a revival of the idea of using the integral equation technique with the singularity points P placed off the boundary. Van Oortmensen of MARIN (then NSMB) developed this approach in the early 1970's for linear water wave diffraction problems; but I believe that the difficulties of defining the optimum location of the points P for complex bodies led to the discontinuation of this work. I wonder whether these difficulties would not be even greater in the study of the interation between a non-linear wave and a surface piercing body.

Cao, Schultz & Beck: Our limited experience indicates that there is a broad optimum range of Z_a for which good solutions can be achieved with little sensitivity to the change in Z_a. We would not expect greater difficulties using non-singular formulations than using singular formulations in the study of the interaction between a nonlinear wave and a surface piercing body. In our opinion, the difficulties of the intersection problem mainly come from the problem itself (for example, the discontinuity of the solution at the intersection either physically or mathematically). Nevertheless, we will have to face these difficulties.

Nakos: It is very worthwhile looking carefully into different discrete formulations of the BVP you consider. And I would like to congratulate the authors for their contribution. A question that arises, however, is the well-conditioning of the matrix as the singular and control points move apart from each other. One expects the condition number to have a minimum when the sources are placed on the boundary. What are your comments about the errors due to the inversion of an ill-conditioned matrix and may that be an explanation of the convergence studies shown in the presentation?

Cao, Schultz & Beck: The reply to P.A. Martin should also answer your question. In addition, our numerical tests show negligible differences between double and single precision computations for the acceptable range of Z_a.